Lecture:
Transmission Lines and Waveguides

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This Lecture

- Introduction to Various Transmission Lines
- Coaxial Lines
  - Wave Impedance
  - Conditions for minimum Damping, maximum Voltage rating, and maximum Power Transmission
  - Attenuation and Power Capability, what are the Technical Limits?
  - Bandwidth → Higher Order Mode TE_{11}–mode Cutoff Frequency
- Waveguides (Round and Rectangular)
  - Most derivations are now in Appendix including full Set of RF field Components
  - Cutoff Frequencies
  - Group and Phase Velocity
  - Examples of Mode Pattern
  - Attenuation of Fields below Cutoff Frequency
  - Poynting Vector
  - Derivation of Transmitted Power in TE_{10} mode of Rectangular Waveguide
- USAPS Experiment with Rectangular Waveguide

Introduction

- Transmission lines and waveguides are utilized to transfer electromagnetic waves carrying energy and information from a source to a receiver
- For an efficient transport one likes to guide the energy inside a line instead of spreading it out in space
- Choice of the line technology depends on the purpose, e.g. **operating frequency range,** the transmitted **power level**, and what **power losses** one can tolerate

Some typical transmission lines

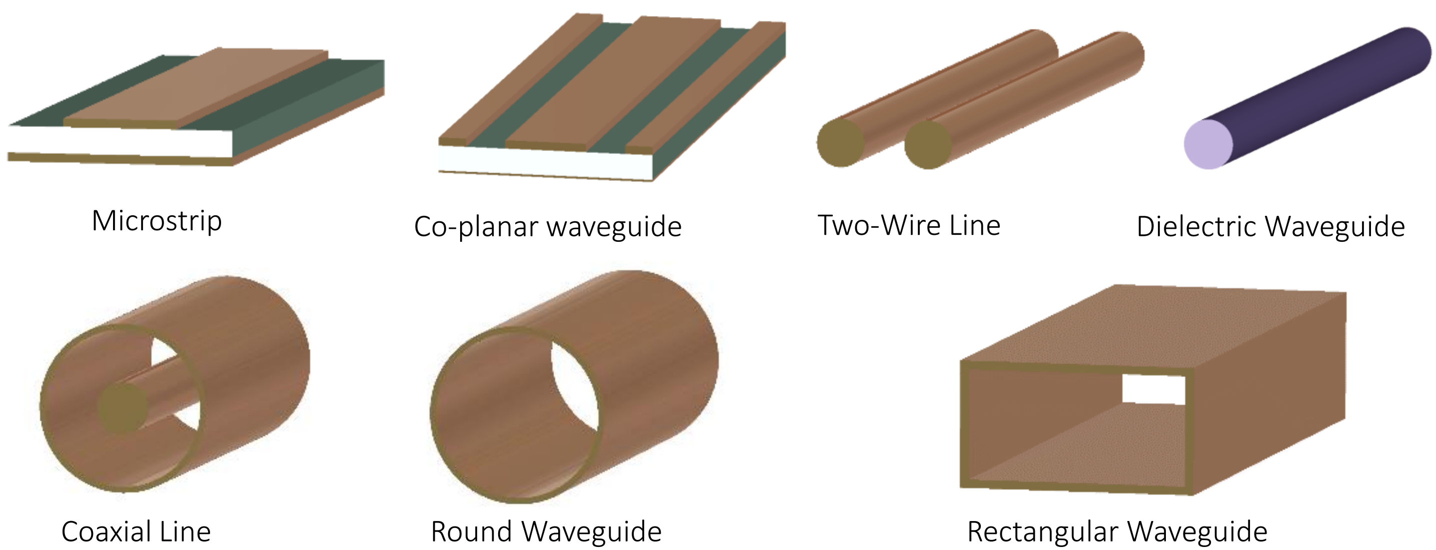
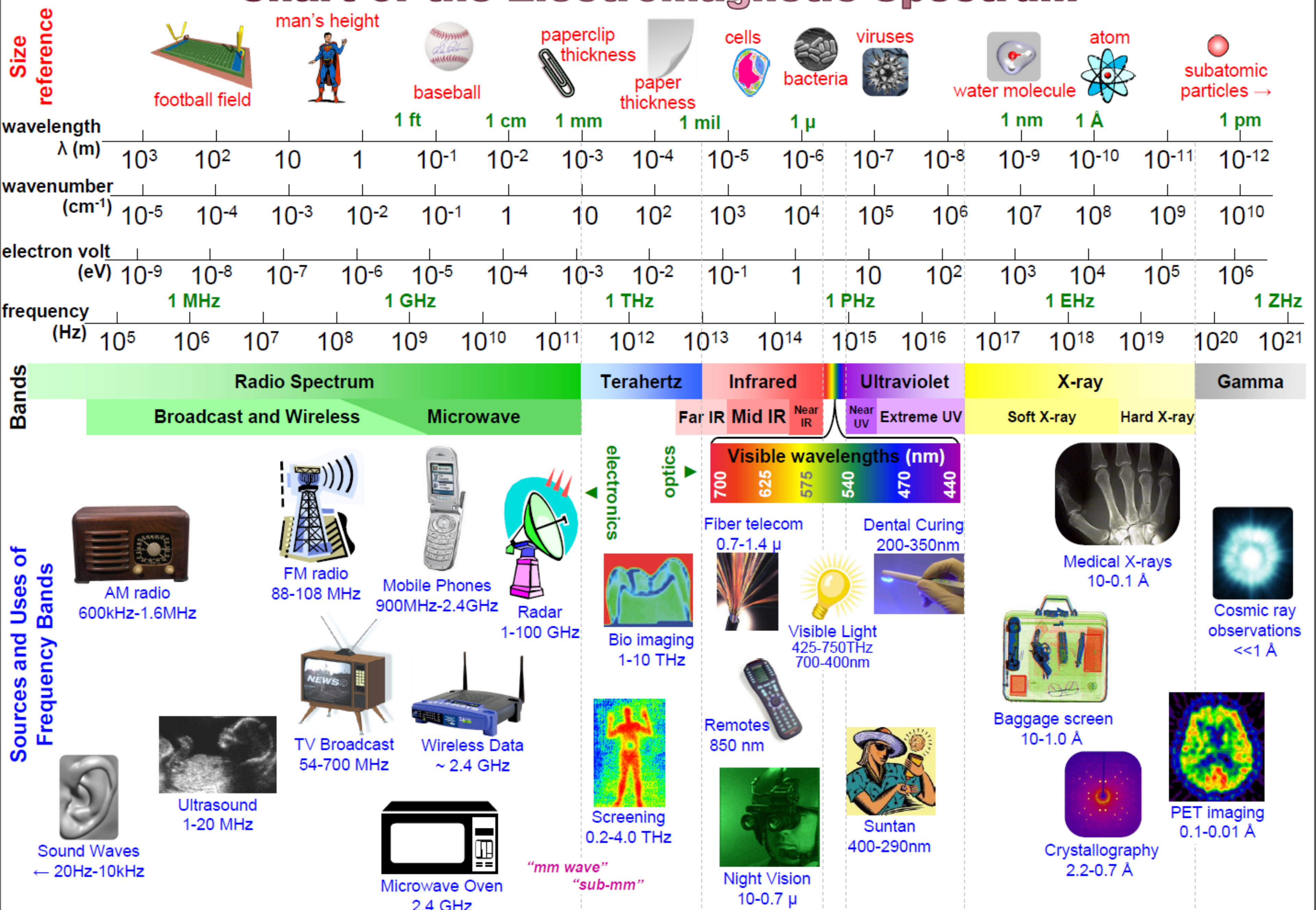


Chart of the Electromagnetic Spectrum



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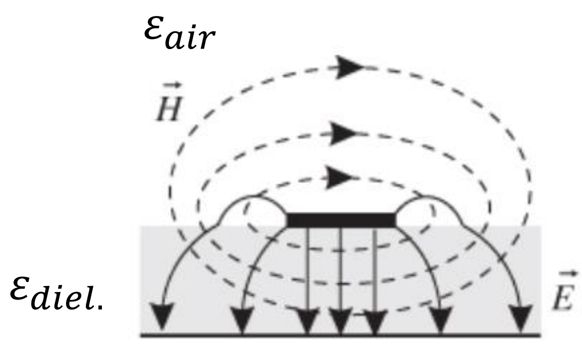
Microstrip Line

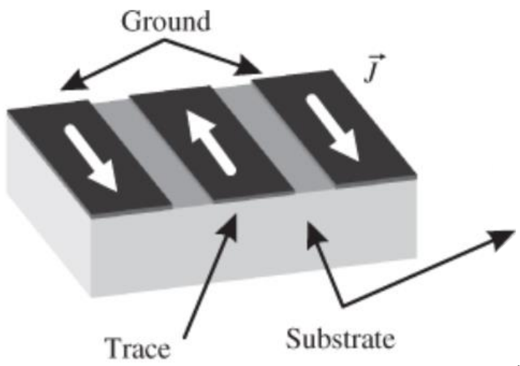
– Microstrip lines are types of planar transmission lines widely used in printed circuit boards (PCBs)
  - Made by a strip conductor, dielectric substrate, and a ground plate
  - Used in the microwave range with typical maximum frequency of 110 GHz
  - Wave is confined mostly in dielectric, but is partially in upper substrate (usually air)
  - The dielectric constant of the substrate usually decreases with frequency as dipolar polarization in the material cannot follow anymore the oscillations of the electric field (starting around 10 GHz)
  - The dielectric constant then approaches more and more that of air if the frequency increases
  - At low frequencies, the fields resemble closely a *TEM* mode ($v = c_0/\sqrt{\varepsilon_r}$) with fields confined in the dielectric, but at high frequency there are more non-negligible longitudinal components of both *E* or *H* resulting in a 'quasi' TEM mode
  - Comparably lossy
  - Not shielded, may radiate parasitically and is vulnerable to cross-talk

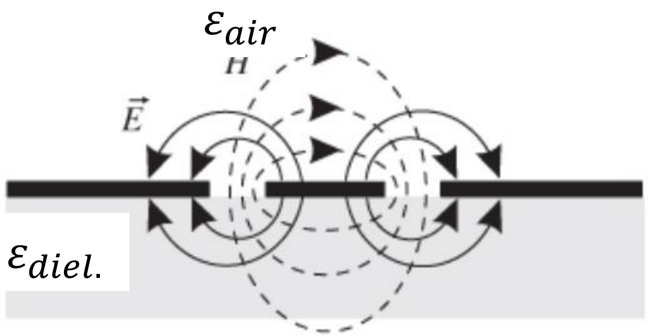


Frank Gustrau, "RF and Microwave Engineering: Fundamentals of Wireless Communications', ISBN: 9781118349571, 2012

Coplanar Waveguide

- Coplanar waveguides (CPWs) are similar to microstrip lines and also used for PCBs
  - Invented later than microstrips (1969 versus 1952)
  - Easier to fabricate since having the return and main conductors in the same plane
  - May or may not be grounded at the bottom
  - Also operate in a quasi-TEM mode at a typical maximum frequency of 110 GHz.





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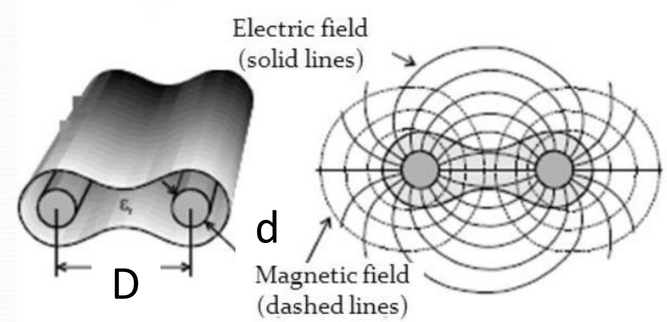
Optical Fibers

- Dielectric waveguides can be optical fibers that have a circular cross-section
  - Consist of a dielectric material surrounded by another dielectric material
  - Allows transmitting optical and infrared signals with small losses (~0.2 dB per 1 km)
  - Power transmitted is in the mW range.

Two-Wire Line

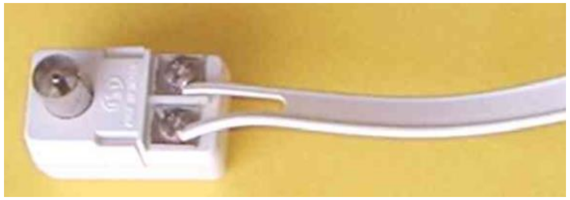
– Two-wire (twin-lead) lines are used for telecommunication to transport RF wave
  - Used e.g. for antenna lines to TV
  - Separation of the wire is small compared to the wavelength (at 30 MHz wavelength is 100 m)
  - Wave is transported in a TEM mode
  - May offer smaller losses in the VHF band than miniature coaxial cables, e.g. 0.55 dB/100m versus 6.6 dB/100m for RG-58
  - However, more vulnerable to interference even if shielded.



$$Z \approx 276\,\Omega\, Log\left(\frac{2D}{d}\right)$$

For D >> d

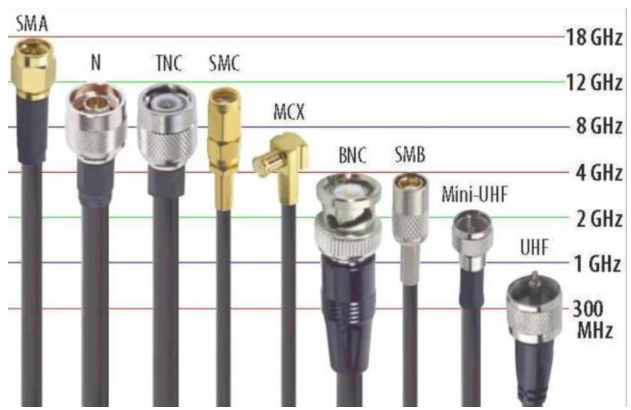
Source: Electromagnetic Waves and Applications Part III, Y. MA



Source: Wikipedia

Coaxial Line

- Coaxial cables are widely used in laboratories and carry signals in the TEM mode..
  - At higher frequencies, the dimensions of the cables should be however limited as higher order modes (with a cutoff) can propagates
  - This in turn limits the power capability
  - Coaxial cables are typically utilized below 3 GHz with attenuation losses of a few dB/100m in the UHF range (around 100 MHz)
  - Losses however quickly rise with frequency (for small cables to ~10 dB/100m at 1 GHz) with an average power rating around just 1kW.
  - The main losses arise due to the skin effect in the inner conductor, which is technically more difficult to cool than the outer conductor
  - At higher frequencies (around 10 GHz) the dielectric losses of the insulator can become dominant
  - By enlarging the coaxial lines diameters (several inches for outer diameter), the power capability may rise above 100 kW (at few hundred MHz) and into the MW regime (at few 10 MHz) with small attenuation losses (< 1dB/100m)

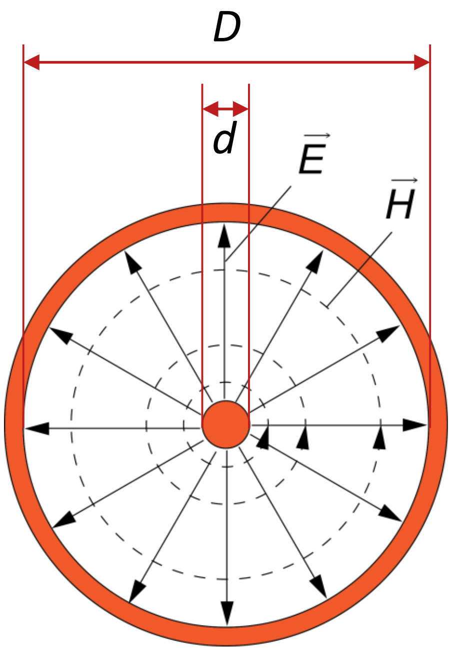


http://www.winpoint.com.tw

Coaxial Line – Wave Impedance

$$Z = \frac{1}{2\pi \cdot \sqrt{\varepsilon_r}} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot Ln\left(\frac{D}{d}\right) \rightarrow Z \approx \frac{60\ \Omega}{\sqrt{\varepsilon_r}} \cdot Ln\left(\frac{D}{d}\right)$$

; recall $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi\ \Omega$



TEM-mode

– *Z* at some ratios D/d

Z (Ω)·$\sqrt{\varepsilon_r}$	D/d
41.56	2
50	2.302
75	3.493

– Last 2 values are common cable impedances, why ?

Coaxial Line – Minimum Damping

- There are attenuation losses along the coaxial line (conduction and dielectric losses)

- **What is the optimum ratio *D/d* to minimize losses in the coaxial cable?**

- Attenuation constant α

$$\alpha = \frac{1}{2\cdot Z}\cdot\frac{1}{\pi}\cdot\left(\frac{1}{d}+\frac{1}{D}\right)\cdot\sqrt{\frac{\mu\omega}{2\sigma}} + \pi f\frac{\sqrt{\varepsilon_r}}{c_0}\cdot tan\delta$$

Resistive losses + dielectric losses

; recall surface resistance $R_S = \sqrt{\frac{\mu\omega}{2\sigma}}$

$tan\delta$ loss tangent of dielectric material

- The 2nd term does not depend on the ratio *D/d*

- We then need to see for which ratio *D/d* the 1st term is at a minimum:

$$\alpha = \frac{2\pi\cdot\sqrt{\varepsilon_r}}{2\cdot Z_0\cdot Ln\left(\frac{D}{d}\right)}\cdot\frac{1}{\pi}\cdot\left(\frac{1}{d}+\frac{1}{D}\right)\cdot R_S \quad ; Z = \frac{1}{2\pi\cdot\sqrt{\varepsilon_r}}\cdot\sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot Ln\left(\frac{D}{d}\right)$$

$$\alpha = \frac{\sqrt{\varepsilon_r}}{Z_0\cdot Ln\left(\frac{\boldsymbol{D}}{\boldsymbol{d}}\right)}\cdot\frac{\boldsymbol{1}}{\boldsymbol{D}}\cdot\left(\frac{\boldsymbol{D}}{\boldsymbol{d}}+1\right)\cdot R_S$$

or

$$\frac{\alpha\cdot Z_0\,\boldsymbol{D}}{\sqrt{\varepsilon_r}\;R_S} = \frac{\frac{\boldsymbol{D}}{\boldsymbol{d}}+1}{Ln\left(\frac{\boldsymbol{D}}{\boldsymbol{d}}\right)} \equiv f_\alpha(D/d)$$

- Note that there is *D* left, not only *D/d*

- One then may ask what is the optimum ratio *D/d*
to achieve the minimum attenuation **at a given diameter of the cable D** ?

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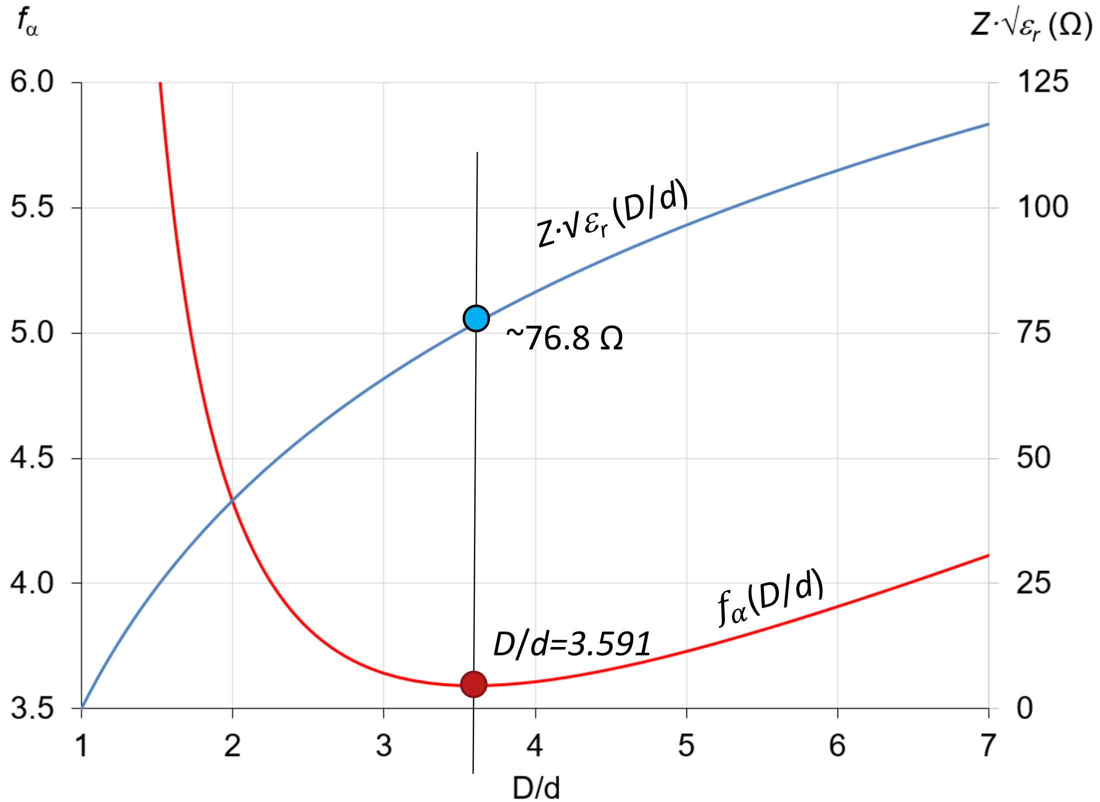
Coaxial Line – Minimum Damping

- Minimum Attenuation: $\boxed{Z_{opt.} \approx \frac{\mathbf{76.8\ \Omega}}{\sqrt{\epsilon_r}}}$ at $\boxed{\frac{D}{d} \approx 3.591}$

- E.g. PTFE with ϵ_r = 2.1 → $Z_{opt.\alpha}$ ~ 53 Ω

- If inner and outer conductor are of different materials, this is not true anymore since conductivity values are different, e.g. Al (D) and Cu (d) then $Z_{opt,\alpha}$ ~ 95 Ω/m



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Coaxial Line
– Maximum Voltage and Maximum Power -

- However, sometimes one rather aims for the **maximum voltage (*V_{max}*) at a given D to avoid a premature RF cable breakdown,** which is given by the dielectric strength of the material (at RF breakdown the dielectric fails to insulate)
- In that case, one wants to **choose *D/d* to minimize the electrical field at given D and given voltage V** between the inner and outer conductor

- In a similar fashion, one finds that $\boxed{Z_{\text{opt. } V_{max}} \sim \frac{\mathbf{60\ \Omega}}{\sqrt{\epsilon_r}}}$ at $\boxed{\frac{D}{d} = 2.718}$

- Moreover, in other cases one desires the **maximum power transmission at a given D**

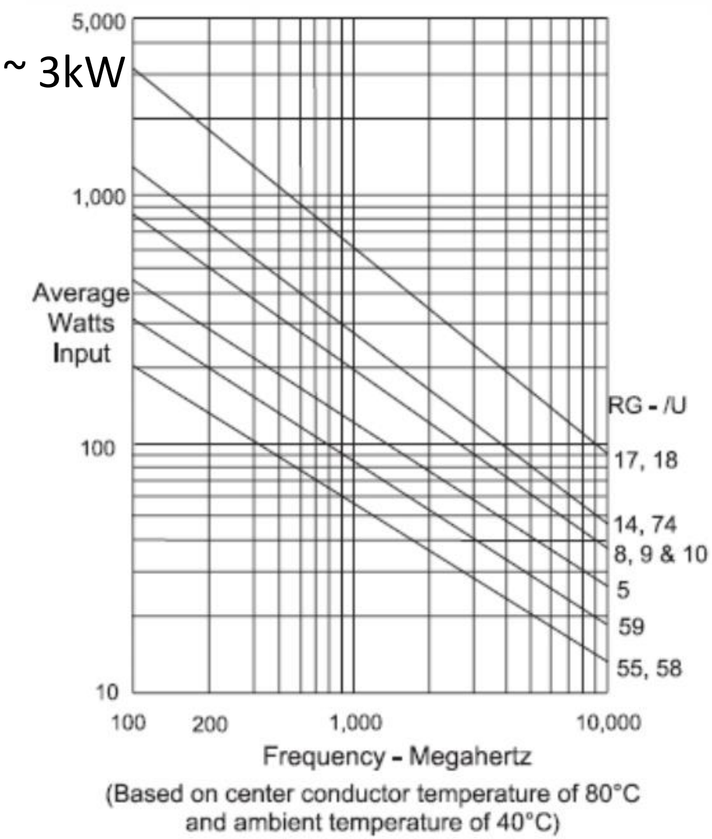
- In this case $\boxed{Z_{\text{opt. } P_{max}} \sim \frac{\mathbf{30\ \Omega}}{\sqrt{\epsilon_r}}}$ at $\boxed{\frac{D}{d} = 1.65}$

Coaxial Line – Power Capability

- Losses quickly rise with frequency

RG=Radio Guide cables

~ 3kW



- To maximize power capability, use biggest cables diameter

$$\alpha = \frac{\sqrt{\varepsilon_r}}{Z_0 \cdot Ln\left(\frac{D}{d}\right)} \cdot \frac{\mathbf{1}}{\boldsymbol{D}} \cdot \left(\frac{D}{d} + 1\right) \cdot R_S$$

RADIO FREQUENCY SYSTEMS
The Clear Choice®

9" HELIFLEX® Air Dielectric Coaxial Cable

HELIFLEX **are air-dielectric cables**. The inner conductor is centered by using a dielectric helix made from high density polyethylene)

Z = 50 Ω

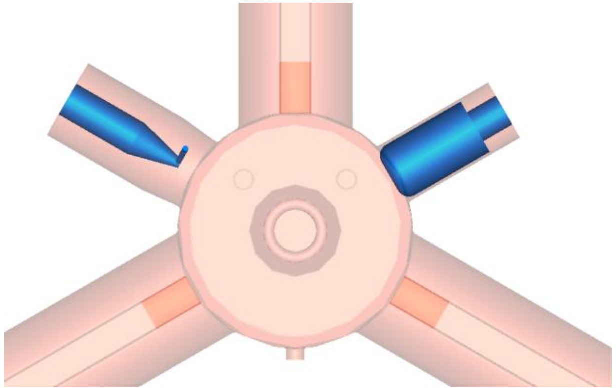
D = 24.77 cm (corrugated Aluminium)
d = 9.94 cm (corrugated Copper tube)

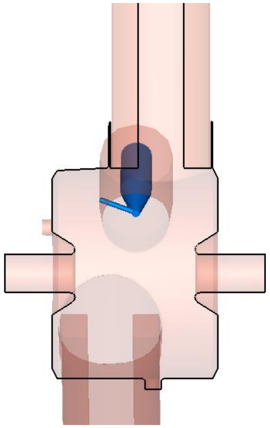
Max. peak power rating: 5.8 MW at 0.5 MHz

but only 236 W at 560 MHz (f_{max})

High Power Coaxial Lines for Cavities

- Consider: 3rd generation storage ring light sources can store few hundreds of mA





500 MHz BESSY (European)
HOM-Damped Cavity

- Powered by coaxial coupler feeding cavity via water-cooled loop coupler
- Typical effective operating voltage is 1 MV
- At I_b = 100 mA, the forward power required for the beam (beam loading) is **100 kW** (CW) at **500 MHz**
- We need more power than shown so far!

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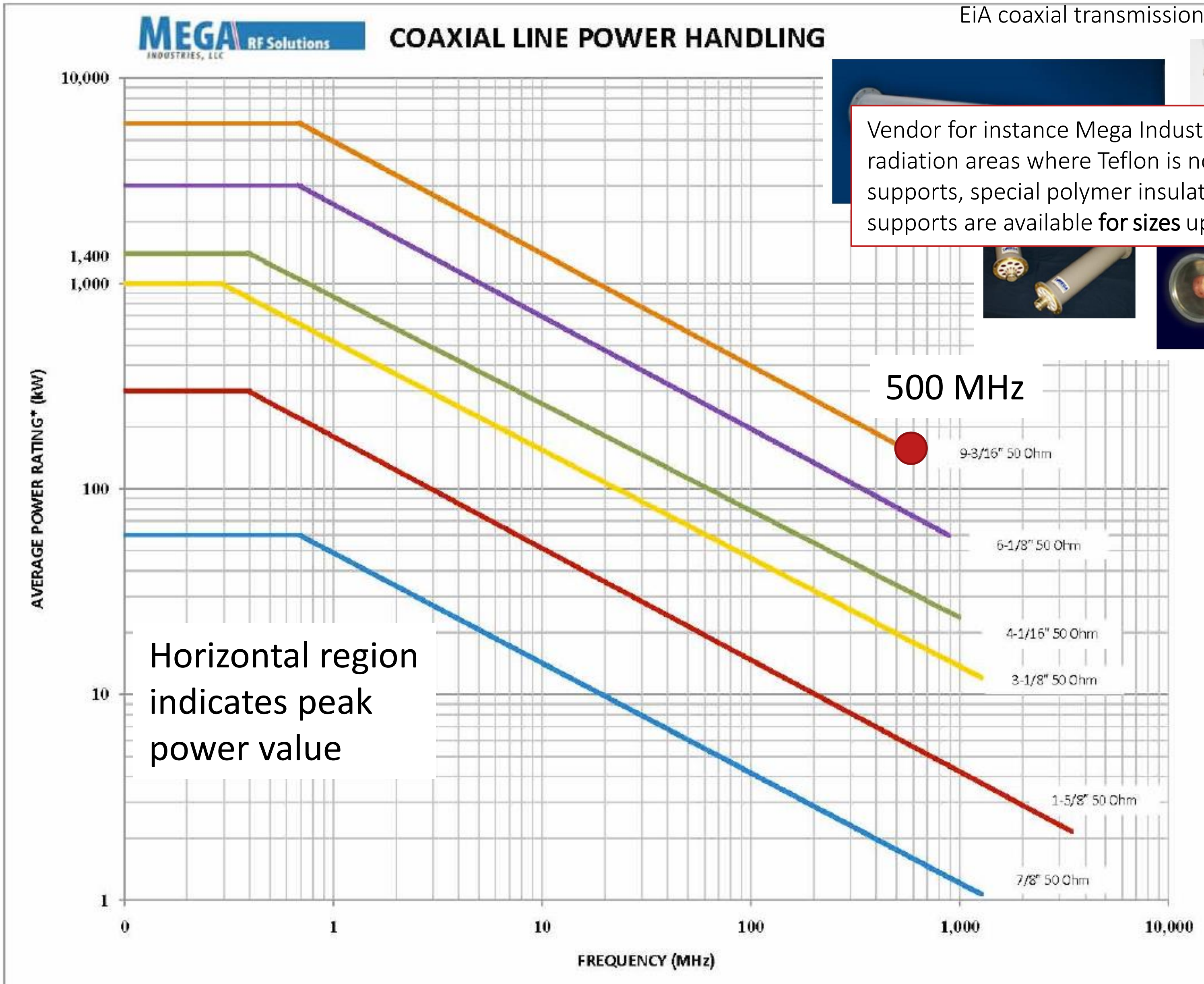
High Power Coaxial Lines for Cavities

EiA coaxial transmission lines (50, 75, or 100 Ω)

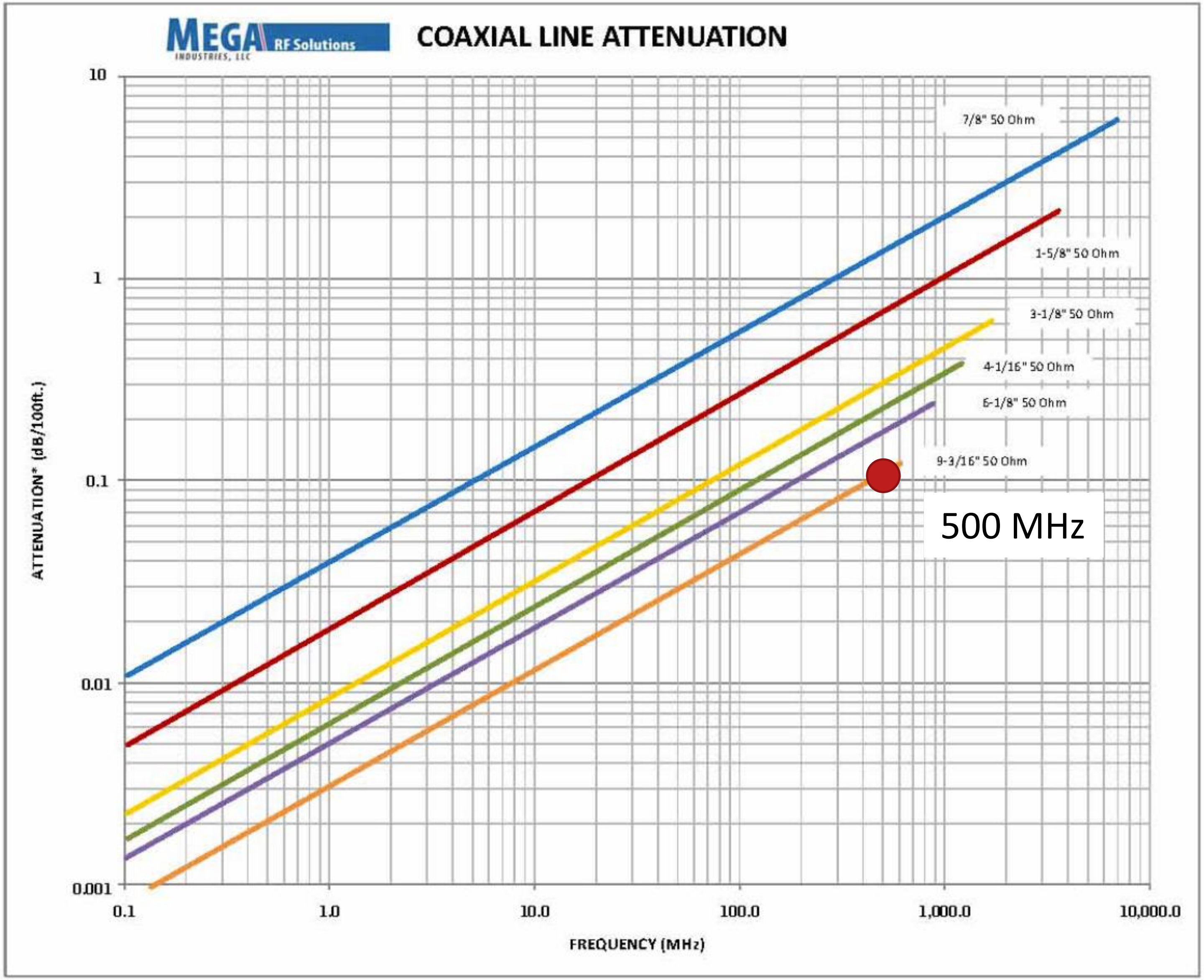


Vendor for instance Mega Industries: In **high** radiation areas where Teflon is not suitable as supports, special polymer insulators or ceramic supports are available **for sizes** up to 14” in diameter.

500 MHz

Horizontal region indicates peak power value

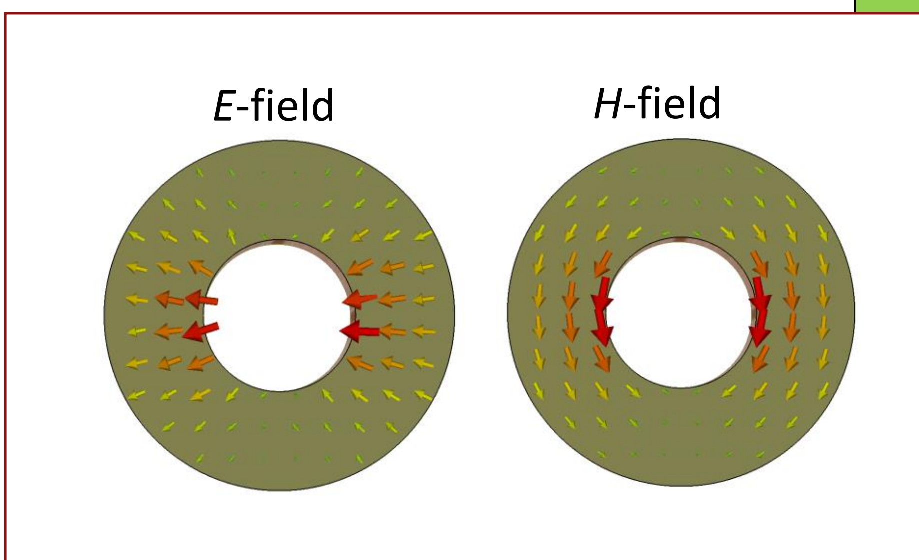
Attenuation in such Coaxial Lines



https://www.megaind.com

What Defines Bandwidth of Transmission Lines?

- To maximize power capability, use biggest cables diameter AND avoid exciting the next mode
- One does not want any higher order mode to propagate beside the lowest (dominant) mode
- In coaxial line the *TEM*-mode is the dominant mode
- Such modes have no cutoff frequency (transmission line works all the way down to DC)
- **Example**: 2nd mode in coaxial cable is a dipole ***TE*$_{11}$-mode**
- This dipole mode changes polarity twice around cable circumference
- The corresponding wavelength equals the cable circumference C, but at which radius?
- **Approximation**: Use average circumference $$\lambda_c^{TE_{11}} = C = \pi \cdot \left(\frac{d+D}{2}\right)$$



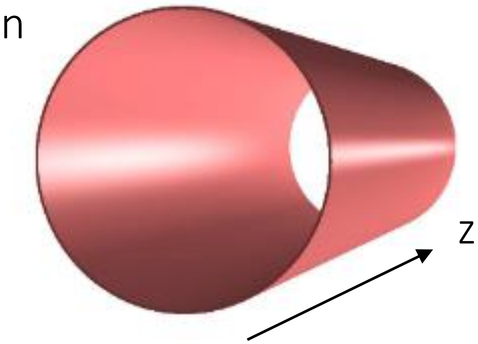
- toff wavelength below which TE_{11} mode can propagate

$$E_{11} = \frac{v}{\lambda_c^{TE_{11}}} = \frac{c_0}{\sqrt{\mu_r \cdot \varepsilon_r}} \cdot \frac{1}{\pi \cdot \left(\frac{d+D}{2}\right)}$$

- toff frequency above which TE_{11} mode can propagate
- acuum) → $f_c^{TE_{11}} = 5.78\ GHz$
-

Round Waveguide

- Derivation of field components in round waveguide follows the same method covered in lecture about resonators
- This time however we have no reflection plate, and wave can propagate freely
- We thus lose one constraint (index p) in longitudinal direction that we had derived for the cylindrical resonator, and only have to deal with two integers m and n
- Appendix covers derivations
- For a perfect conductor (no resistive attenuation) we obtain the propagation constants for *TE*- and *TM*-modes



$$\boxed{\gamma = i\beta = \pm\, i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x_{mn}\ or\ x'_{mn}}{R}\right)^2}}$$

x_{mn} for *TM*-modes
x'_{mn} for *TE*-modes

- We see that only if β is real, the wave can propagate without decay by means of $e^{-i\beta z}$
- This leads to so-called **cutoff frequencies**, above which wave may propagate for given m, n

$$\mu\epsilon\omega^2 \geq \left(\frac{x_{mn}\ or\ x'_{mn}}{R}\right)^2 \longrightarrow \boxed{\omega_c = \frac{1}{\sqrt{\mu\epsilon}} \cdot \left(\frac{x_{mn}\ or\ x'_{mn}}{R}\right)}$$

- The spectrum of possible modes above cutoff frequencies is continuous
- Inserting last in first equation on this page yields: $\boxed{\beta^2 = \mu\epsilon(\omega^2 - \omega_c^2)}$; *TE* or *TM* modes

Phase and Group Velocity in Waveguide

$$v_{ph} = \frac{\omega}{k}$$

Phase velocity ; in free space $v_{ph} = c_0$ due to ; $k^2 = \mu\epsilon\omega^2$

- In a waveguide the wavenumber is constrained compared to free space ($k = \sqrt{\mu\epsilon}\omega$) due to $\beta^2 = \mu\epsilon(\omega^2 - \omega_c^2)$

$$\beta = k_z = \frac{2\pi}{\Lambda} = \sqrt{k^2 - {k_c}^2} = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2} \longrightarrow \omega = \sqrt{\frac{{k_z}^2}{\mu\epsilon} + \omega_c^2}$$

- Λ is the wavelength of the waveguide ('guide length')

$$v_{ph} = \frac{\omega}{k_z} = f \cdot \Lambda = \frac{\omega}{\sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}} = \frac{\mathrm{v}}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

Phase velocity is always > speed of light

$$v_{gr} = \frac{d\omega}{dk}$$

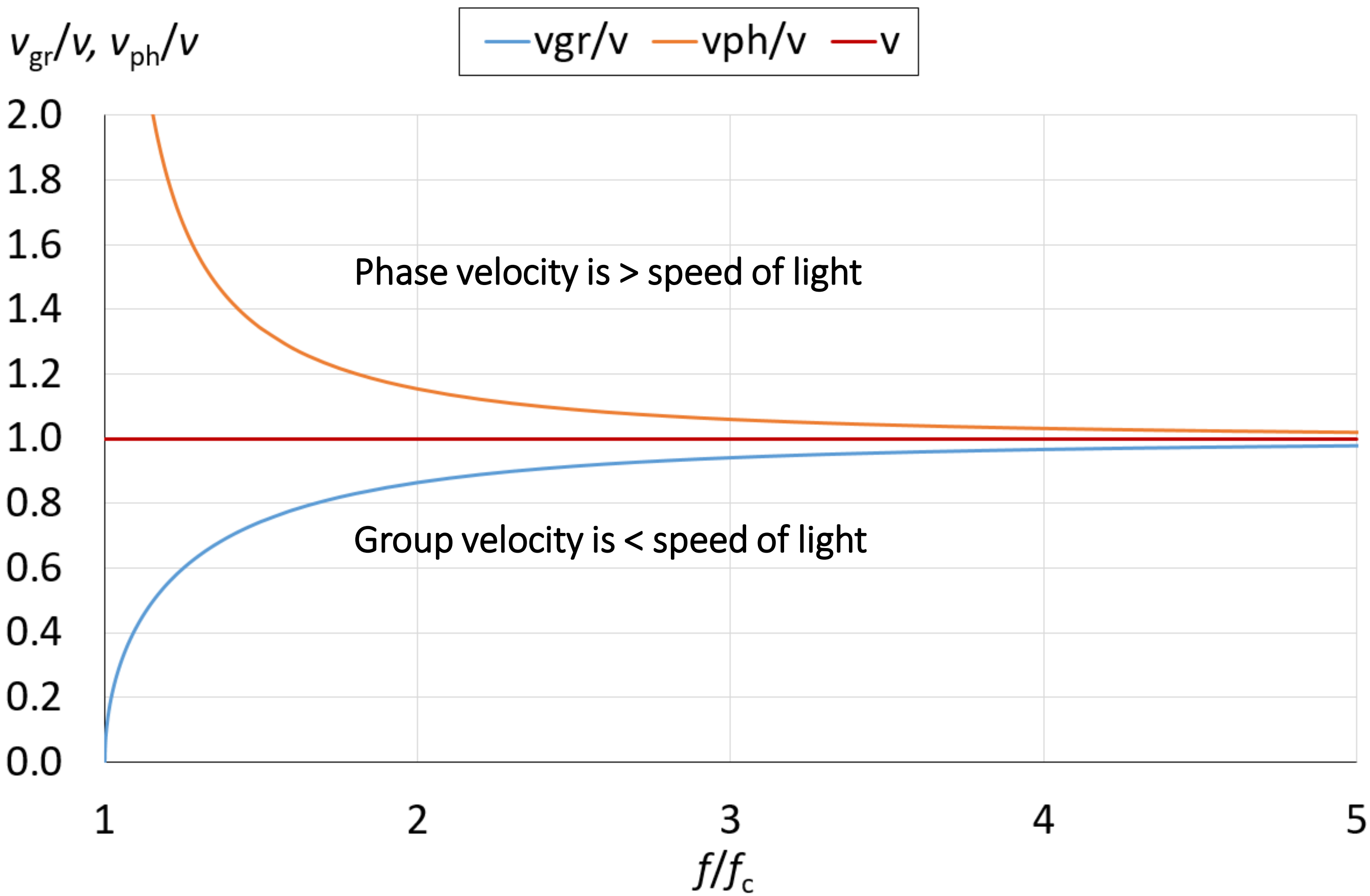
Group velocity (energy flows at group velocity)

$$v_{gr} = \frac{d\omega}{dk} = \frac{k_z}{\mu\epsilon\omega} = \frac{1}{\sqrt{\mu\epsilon}} \frac{\sqrt{\omega^2 - \omega_c^2}}{\omega} = \mathrm{v} \cdot \sqrt{1 - \frac{f_c^2}{f^2}}$$

Group velocity is < speed of light

Group velocity is not constant with frequency (dispersion)

Phase and Group Velocity in Waveguide



Roots of Bessel Function and its Derivative
Lowest Cutoff Frequencies and Degeneracy

$$\omega_c^{TM} = \frac{1}{\sqrt{\mu\epsilon}} \cdot \left(\frac{x_{mn}}{R}\right)$$

$$\omega_c^{TE} = \frac{1}{\sqrt{\mu\epsilon}} \cdot \left(\frac{x'_{mn}}{R}\right)$$

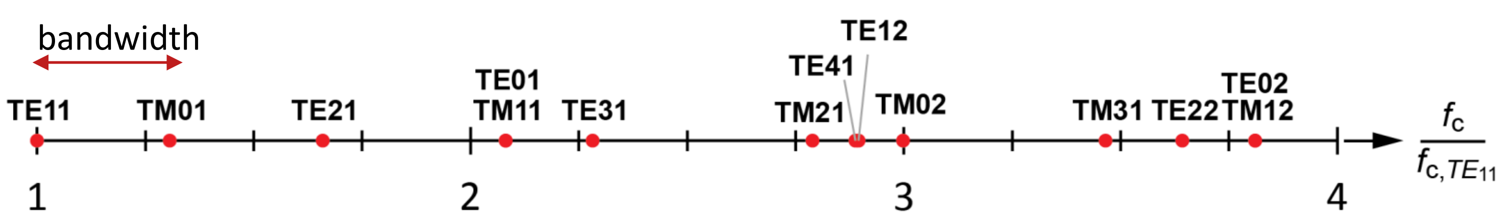
x_{mn}

m \ n	1	2	3	4
0	2.40483	5.52008	8.65373	11.7915
1	3.83171	7.01559	10.1735	13.3237
2	5.13562	8.41724	11.61980	14.79600
3	6.38016	9.76102	13.01520	16.22350

x'_{mn}

m \ n	1	2	3	4
0	3.83171	7.01559	10.1735	13.3237
1	1.84118	5.33144	8.53632	11.7060
2	3.05424	6.70613	9.96947	13.17040
3	4.20119	8.01524	11.34590	14.58580

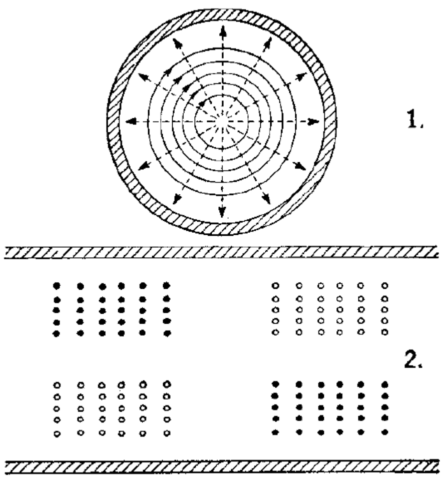
- 1st cutoff is dipole mode TE_{11} ($x'_{11} = 1.84118$)
- 2nd cutoff is monopole mode TM_{01} ($x_{01} = 2.40483$)
- Degeneracy (differing modes, but same cutoff frequency) occurs when $x_{1n} = x'_{0n}$ → $f_{1n}^{\mathrm{TM}} = f_{0n}^{\mathrm{TE}}$
- Recommended operating bandwidth (single-mode operation) is from slightly above TE_{11}-mode cutoff to **maximally** TM_{01}-mode cutoff (factor ~1.31 higher than TE_{11}-mode cutoff)



Cutoff frequencies in a round waveguide in sequential order for the first *TE* and *TM* modes (normalized to first TE_{11} cutoff frequency)

TE-Mode Pattern

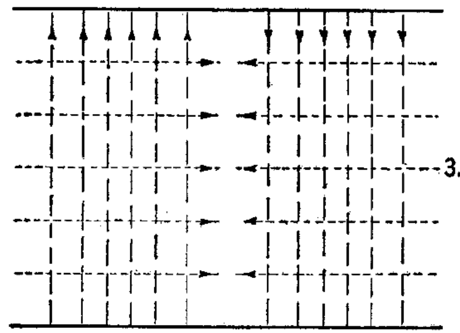
1. Cross-sectional view
2. Longitudinal view through plane l-l
3. Surface view from s-s

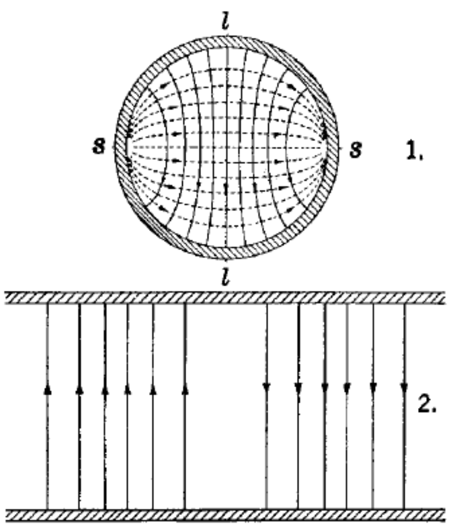


TE_{01}

H_{01}

$\lambda_c = 1.640\ a$

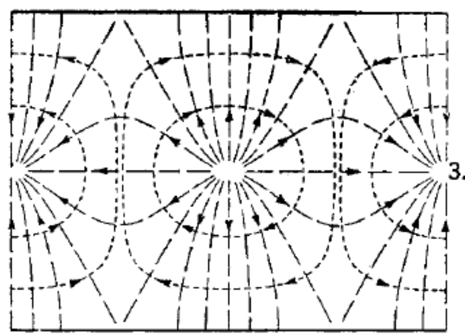




TE_{11}

H_{11}

$\lambda_c = 3.412\ a$



Waveguide and Frequencies below Cutoff

- We recall that for the propagation constant $\boxed{\gamma = i\beta}$ and $\boxed{\beta = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_c^2}}$

- Specific modes (with indices *m*, *n*) propagate undamped (perfect conductor) according to $e^{-i\beta z}$ and only above the cutoff frequency

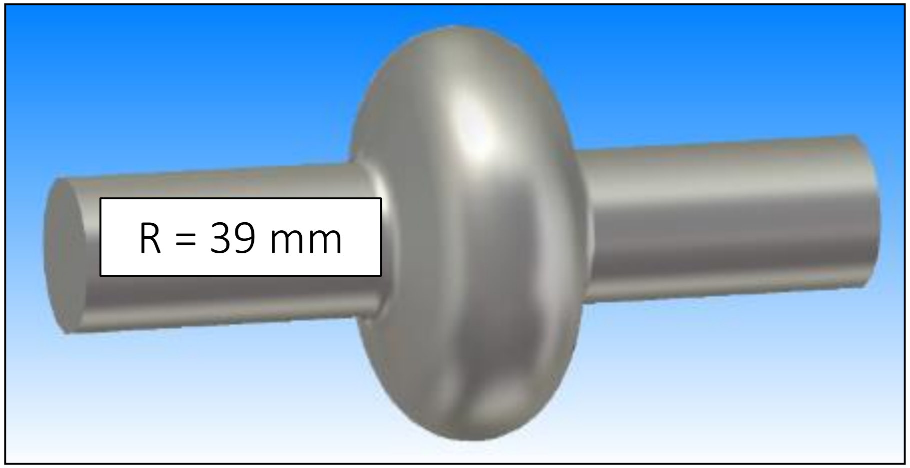
- **What if mode frequency is smaller than its cutoff frequency ?**

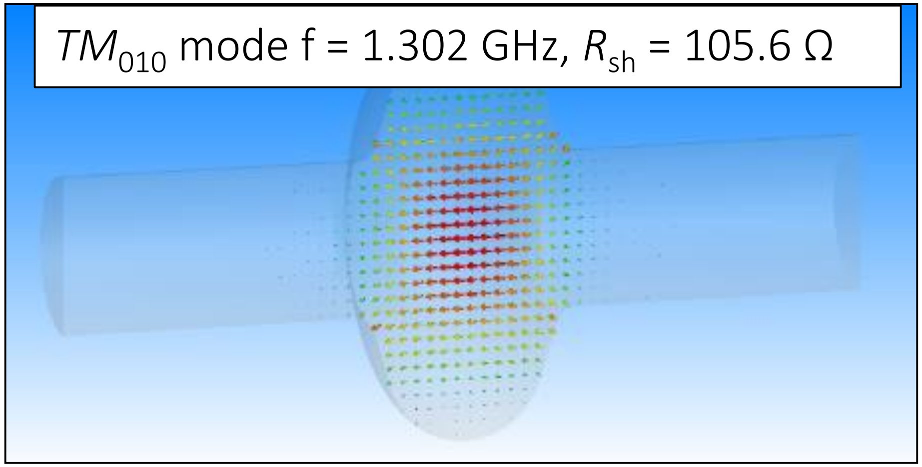
$$\boxed{\beta = i \cdot \sqrt{\mu\epsilon}\sqrt{\omega_c^2 - \omega^2}} \quad ; \text{ for } \omega < \omega_c$$

- β becomes imaginary itself and γ becomes real

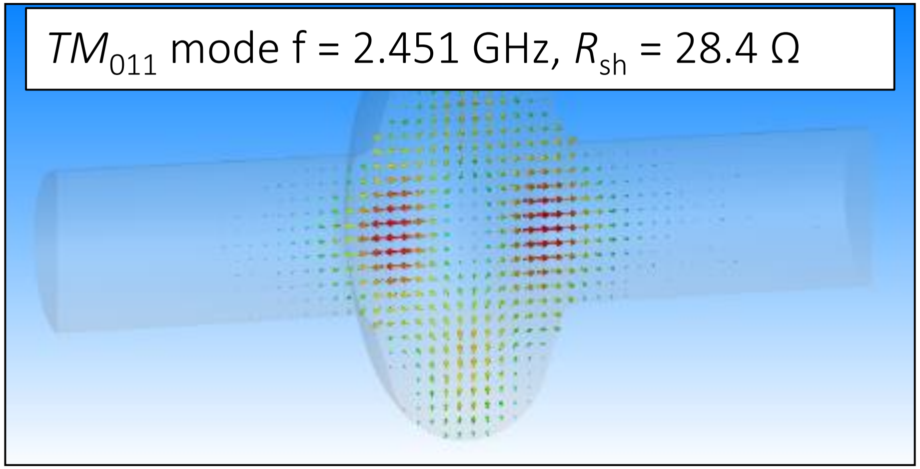
- **The wave is therefore damped even for the loss-less case (α =0) according to $e^{-\gamma z}$**

Example: Single-Cell (TESLA) Cavity





- First monopole harmonic of fundamental mode (TM_{011})
- First **relevant** monopole cutoff frequency is TM_{01} (not TE_{11})



$$f_{c,TM01} = \frac{c}{2\pi} \cdot \frac{x_{mn}}{R} = 2.94\ GHz > f_{TM011}$$

→ TM_{011} resonator mode is trapped inside cavity → we need HOM-couplers to suppress this field

Even More Power: Hollow Waveguides

- Hollow waveguides can transmit very high average power signals in the microwave spectrum
  - Rectangular and round waveguides are commonly employed.
  - Without an inner conductor, they can sustain much higher power levels than coaxial lines
  - Metal walls can be readily cooled
  - Recommended bandwidth is limited – as for coaxial line – by preventing the next waveguide mode to co-exist with the dominant mode
  - The inner surface of the waveguides can be plated with high conductive material (e.g. copper, silver, gold) to reduce losses due to the skin effect
  - Average power levels into the MW range can be achieved with rather small attenuation losses (few dB/100 m) with large scale waveguides
  - Standard rectangular waveguides (WR) sizes are available up to WR2300 (0.584 m (23") x 0.2921 m) covering 320-450 MHz and down to WR3 (0.864 mm x 0.432 mm) covering 220-330 GHz.

- In SRF accelerators we may run 20 MV/m in CW in Energy Recovery Linacs with large currents (up to 1A machines have been proposed in the past)
- The injector of an ERL is not energy-recovered → If structure is 1m long, we have a voltage of 20 MV (CW) → beam loading at 100 mA is then already 2 MW.
- This is the power we would need to deliver into cavity (wall losses are negligible)
- Otherwise or we need to split powerand/or make cavities shorter to reduce require power per cavity

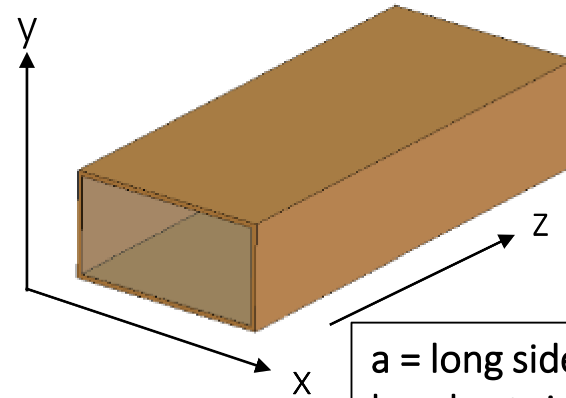
Rectangular Waveguide
Cutoff Frequency, Phase and Group Velocity

- The cutoff frequency can be calculated by (no full derivation here, refer to textbooks):

$$f_c = \frac{v}{2} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

; v is speed of light in linear medium

$$\omega_c = v \cdot \sqrt{\left(\frac{m \cdot \pi}{a}\right)^2 + \left(\frac{n \cdot \pi}{b}\right)^2}$$



a = long side
b = short side

- We obtain the same relations for the phase and group velocity with the given cutoff frequency in the rectangular waveguide as for round waveguides and all other related consequences apply similarly

$$v_{ph} = \frac{\omega}{k} = \frac{v}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

Phase velocity

$$v_{gr} = \frac{d\omega}{dk} = v\sqrt{1 - \frac{f_c^2}{f^2}}$$

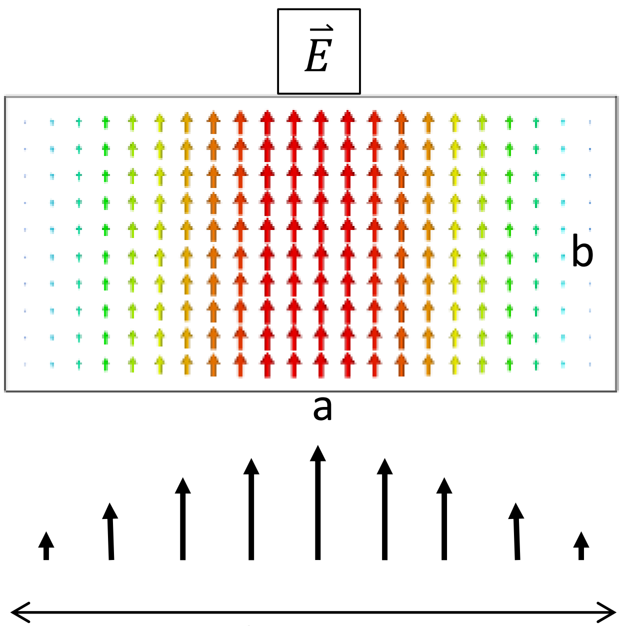
Group velocity

Rectangular Waveguide
Cutoff Frequencies of Lowest Modes

$$f_c = \frac{v}{2} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- **Note: For *TE*-modes: m,n ≥ 0, but m=n=0 is not allowed since only trivial solution exist**
- **Note: For *TM*-modes: m,n ≥ 1**
- → 1st dominant mode is dipole TE_{10}-mode (m = 1, n= 0)
- Cutoff frequency can be easily derived graphically

TE_{10}



$a = \lambda/2$

$$\lambda_c^{TE_{10}} = 2 \cdot a$$

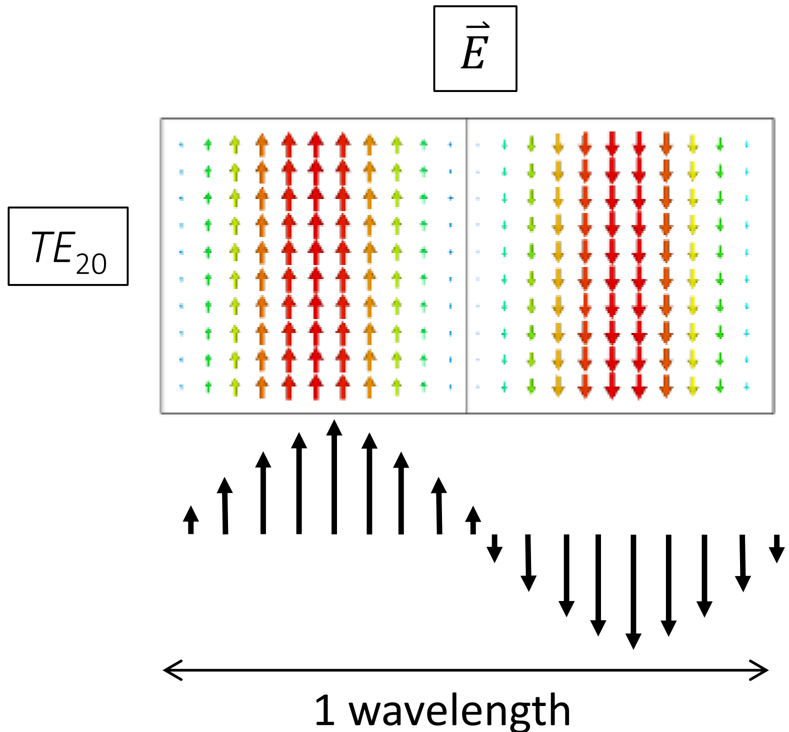
$$f_c^{TE_{10}} = \frac{v}{2 \cdot a}$$

Rectangular Waveguide
Cutoff Frequencies of Lowest Modes

$$f_c = \frac{v}{2} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- Note: For *TE* modes: m,n ≥ 0, but m=n=0 is not allowed since only trivial solution exist
- Note: For *TM* modes: m,n ≥ 1

- Ditto for the TE_{20}-mode



$a = \lambda$

$\lambda_c^{TE_{20}} = a$

$$f_c^{TE_{20}} = \frac{v}{a}$$

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Rectangular Waveguide
Cutoff Frequencies of Lowest Modes

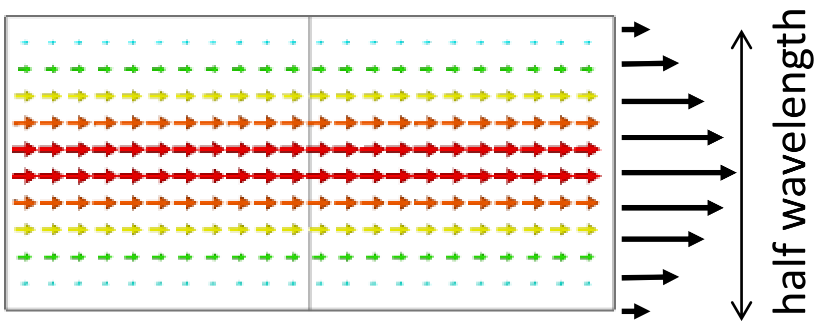
$$f_c = \frac{v}{2} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- Note: For *TE* modes: m,n ≥ 0, but m=n=0 is not allowed since only trivial solution exist
- Note: For *TM* modes: m,n ≥ 1

  - Ditto for the TE_{01}-mode

$\vec{E}$

TE_{01}



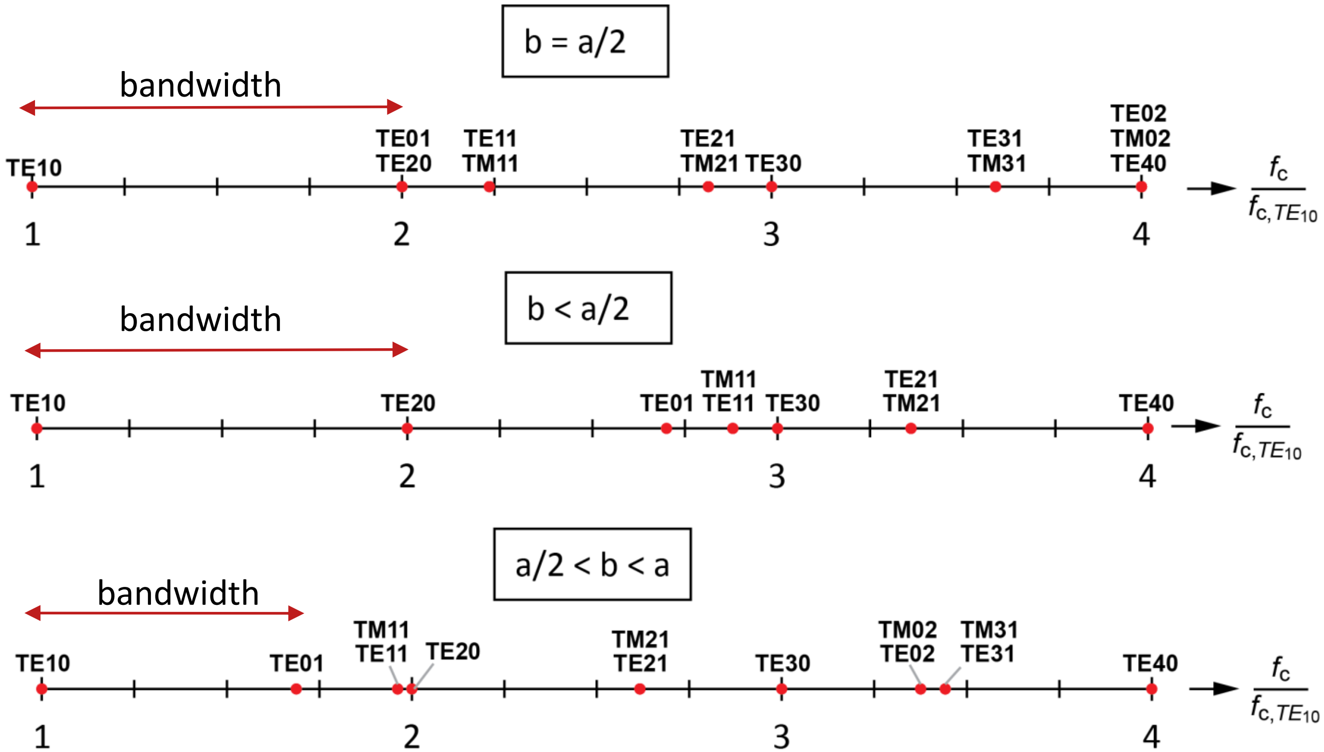
$b = \lambda/2$

$\lambda_c^{TE_{01}} = 2 \cdot b$

$$\boxed{f_c^{TE_{01}} = \frac{v}{2 \cdot b}}$$

Lowest Cutoff Frequencies and Degeneracy

- Various mode degeneracies may occur depending on ratio of waveguide height (b) to width (a) and for all *TE* and *TM* modes with same indices m and n
- Again: **Recommended operating bandwidth (single-mode operation)** is from slightly above TE_{10}-mode cutoff to **maximally** TE_{20}-mode cutoff for b ≤ a/2 (larger bandwidth than for round waveguides). For b > a/2 the bandwidth is reduced as TE_{01}-mode becomes 2nd mode



Cutoff frequencies in a rectangular waveguide for the first *TE* and *TM* modes (normalized to first TE_{10} cutoff frequency) depending on the ratio of the waveguide height to the waveguide width b. Mode degeneracies may occur.

Power Transmission in a Rectangular Waveguide

Power Transmission along a Waveguide

- Power transmitted along a waveguide can be generally by integrating the Poynting vector over the-cross section of the waveguide

$$\vec{S} = \vec{E} \times \vec{H}$$ **Poynting vector**

- Poynting vector points in the direction of the wave propagation and is the energy transferred per unit area and per unit time (units are V/m· A/m = W/m^2)

- For harmonic signals, the time-averaged power is given by real part of cross product integrated over the transverse cross-section of the guide normal to the propagation

$$P_{avg} = \frac{1}{2} \int_{dS} Re\{E_{trans} \times H^*{}_{trans}\} \cdot \hat{n} \cdot dS$$

- **Note that ratio of the transverse components *E*$_{trans}$/*H*$_{trans}$ determines the wave impedance *Z***

$$\longrightarrow \quad (E_{trans} \times H^*{}_{trans}) \cdot \hat{n} = E_{trans} \cdot H^*{}_{trans} = Z \cdot |H_{trans}|^2 = \frac{1}{Z} \cdot |E_{trans}|^2$$

- The transverse components of ***E*** and ***H*** are all normal to each other for *TEM*, *TE*, and *TM* waves and cross product points in direction of $\hat{n} \cdot dS$

- The transmitted power through the waveguide cross-section is thus:

$$P_{avg} = \frac{1}{2} \cdot Z \int_{dS} |H_{trans}|^2 \cdot dS = \frac{1}{Z} \int_{dS} |E_{trans}|^2 \cdot dS$$

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Example: Power Transmitted in TE_{10} mode

- For dominant TE_{10} mode, the non-vanishing transverse electric field component is:

$$E_y(z=0) = -iH_0 \cdot \frac{\omega \cdot \mu}{k_c^2} \cdot \frac{\pi}{a} \cdot sin\left(\frac{\pi \cdot x}{a}\right)$$

- Consequently:

$$P_{avg} = \frac{1}{2} \cdot \frac{1}{Z_{wave}} \int_{dS} |E_{trans}|^2 \cdot dS =$$

$$= \frac{1}{2} \cdot \frac{1}{Z_{wave}} \cdot \int_{y=0}^{b} dy \int_{x=0}^{a} dx \cdot \left(H_0 \cdot \frac{\omega \cdot \mu}{k_c^2} \cdot \frac{\pi}{a} \cdot sin\left(\frac{\pi \cdot x}{a}\right)\right)^2$$

$$= \frac{1}{2} \cdot \frac{1}{Z_{wave}} \cdot b \cdot {H_0}^2 \cdot \left(\frac{\omega \cdot \mu}{\pi} \cdot a\right)^2 \int_{x=0}^{a} dx \cdot sin\left(\frac{\pi \cdot x}{a}\right)^2 \qquad ; k_c^2 = \left(\frac{\pi}{a}\right)^2$$

$$= \frac{1}{2} \cdot \frac{1}{Z_{wave}} \cdot b \cdot {H_0}^2 \cdot \left(\frac{\omega \cdot \mu}{\pi} \cdot a\right)^2 \cdot \left(\frac{x}{2} - \frac{a \cdot \sin(\frac{2\pi \cdot x}{a})}{4\pi}\right)\Bigg|_0^a \qquad ; \int dx\, sin\left(\frac{\pi \cdot x}{a}\right)^2 = \frac{x}{2} - \frac{a \cdot \sin(\frac{2\pi \cdot x}{a})}{4\pi}$$

$$\boxed{P_{avg} = \frac{1}{2} \cdot \frac{1}{Z_{wave}} \cdot b \cdot {H_0}^2 \cdot \left(\frac{\omega \cdot \mu}{\pi} \cdot a\right)^2 \cdot \frac{a}{2}}$$

Example: Wave Impedance TE_{10} mode

- What is yet missing is an expression for the wave impedance *Z*
- We thus need $\boldsymbol{E}_{trans}/\boldsymbol{H}_{trans}$: The only non-vanishing transverse electric and magnetic field components for TE_{10} mode are:

$$E_y(\omega) = -iH_0 \cdot \frac{\omega \cdot \mu}{k_c^2} \cdot \frac{m\pi}{a} \cdot sin\left(\frac{m \cdot \pi \cdot x}{a}\right) \cdot e^{i\omega t}$$

$$H_x(\omega) = iH_0 \cdot \frac{\beta}{k_c^2} \cdot \frac{m\pi}{a} \cdot sin\left(\frac{m \cdot \pi \cdot x}{a}\right) \cdot e^{i\omega t}$$

$$\rightarrow \quad Z_{TE_{10}} = -\frac{E_y(z=0)}{H_x(z=0)} = \frac{\omega \cdot \mu}{\beta} \quad \rightarrow \quad Z_{TE_{10}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \cdot \sqrt{\frac{\mu_r}{\epsilon_r}} \cdot \frac{1}{\sqrt{1 - \frac{f_c{}^2}{f^2}}}$$

$$; \beta = \sqrt{\mu\epsilon}\omega \cdot \sqrt{1 - \frac{f_c{}^2}{f^2}}$$

with $f_{c,TE10} = \frac{\mathrm{v}}{2a}$

- From lecture on Maxwell's equation we remember that vacuum impedance

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi \approx 376.73\ [\Omega] \quad \rightarrow \quad Z_{TE_{10}} = \sqrt{\frac{\mu_r}{\epsilon_r}} \cdot \frac{Z_0}{\sqrt{1 - \frac{f_c{}^2}{f^2}}}$$

- The TE_{10} waveguide impedance is therefore always greater than the free space wave impedance → **Reflection occurs if wave would propagate out into free space**

Example: Power Transmitted in TE_{10} mode

- By inserting the wave impedance we eventually obtain:

$$P_{avg}=\frac{1}{2}\cdot\frac{\beta}{\omega\cdot\mu}\cdot b\cdot {H_0}^2\cdot\left(\frac{\omega\cdot\mu}{\pi}\cdot a\right)^2\cdot\frac{a}{2}=\frac{1}{4}\cdot\frac{\beta\cdot\omega\cdot\mu}{\pi^2}\cdot a^3\cdot b\cdot {H_0}^2 \quad ; Z_{TE_{10}}=\frac{\omega\cdot\mu}{\beta}$$

$$=\frac{\omega\cdot\mu\cdot\sqrt{\mu\epsilon}\omega}{4\pi^2}\cdot a^3\cdot b\cdot {H_0}^2\cdot\sqrt{1-\frac{{f_c}^2}{f^2}} \qquad ; \beta=\sqrt{\mu\epsilon}\omega\cdot\sqrt{1-\frac{{f_c}^2}{f^2}}$$

$$=f^2\cdot\mu\cdot\sqrt{\mu\epsilon}\cdot a^3\cdot b\cdot {H_0}^2\cdot\sqrt{1-\frac{{f_c}^2}{f^2}}$$

$$=f^2\cdot\mu\cdot\frac{1}{\mathrm{v}}\cdot a^3\cdot b\cdot {H_0}^2\cdot\sqrt{1-\frac{{f_c}^2}{f^2}} \qquad ; \mathrm{v}=\frac{1}{\sqrt{\mu\epsilon}}$$

$$=\sqrt{\frac{\mu}{\varepsilon}}\cdot\frac{f^2}{\mathrm{v}^2}\cdot a^3\cdot b\cdot {H_0}^2\cdot\sqrt{1-\frac{{f_c}^2}{f^2}} \qquad ; \mathrm{v}=\frac{1}{\sqrt{\mu\epsilon}}$$

$$\boxed{P_{avg}=\frac{1}{4}\cdot\sqrt{\frac{\mu}{\varepsilon}}\cdot\frac{f^2}{{f_c}^2}\cdot a\cdot b\cdot {H_0}^2\cdot\sqrt{1-\frac{{f_c}^2}{f^2}}} \qquad ; f_{c,TE10}=\frac{\mathrm{v}}{2a}$$

with H_0 the peak field amplitude

Example: Power Limit in TE_{10} mode (air-filled)

- The electrical field amplitude of TE_{10} mode (see appendix) is

$$\boxed{|E_y| = H_0 \cdot \frac{\omega \cdot \mu}{k_c^2} \cdot \frac{\pi}{a} \cdot sin\left(\frac{\pi \cdot x}{a}\right)}$$

- Maximum electric field amplitude is in the center of the waveguide (x = a/2)

$$\boxed{\left|E_y\right| = H_0 \cdot \frac{\omega \cdot \mu}{k_c^2} \cdot \frac{\pi}{a} = H_0 \cdot \frac{\omega \cdot \mu}{\pi} \cdot a} \quad ; k_c^2 = \left(\frac{\pi}{a}\right)^2$$

- If the waveguide is filled with air (1 bar), the **dielectric strength** is 3 MV/m (E_{max})

- **The electric field shall not exceed the dielectric strength**, thus:

$$\boxed{H_{0,max} = \frac{\pi}{\omega \cdot \mu \cdot a} \cdot E_{max}}$$

- Inserting into transmitted power formula yields:

$$\rightarrow \quad \boxed{P_{avg,max} = \frac{1}{4} \cdot \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{f^2}{f_c{}^2} \cdot a \cdot b \cdot \left(\frac{\pi}{\omega \cdot \mu \cdot a}\right)^2 \cdot E_{max}{}^2 \cdot \sqrt{1 - \frac{f_c{}^2}{f^2}}}$$

$$\boxed{P_{avg,max} = \frac{1}{16} \cdot \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{1}{f_c{}^2} \cdot \frac{b}{a} \cdot \frac{1}{\mu^2} \cdot E_{max}{}^2 \cdot \sqrt{1 - \frac{f_c{}^2}{f^2}}}$$

Example: WR650 Waveguide

- Standard WR650 rectangular waveguides (6.5” x 3.25”) with TE_{10} cutoff at ~908 MHz are often used to power L-band accelerators
- For JLab’s cavities (f_{RF} = 1.497 MHz) the maximally transmitted power in air-filled waveguide WR650 waveguides is ~**65 MW** (far above the requirements of ~13 kW (forward power) for upgrade cavities)
- What about attenuation?

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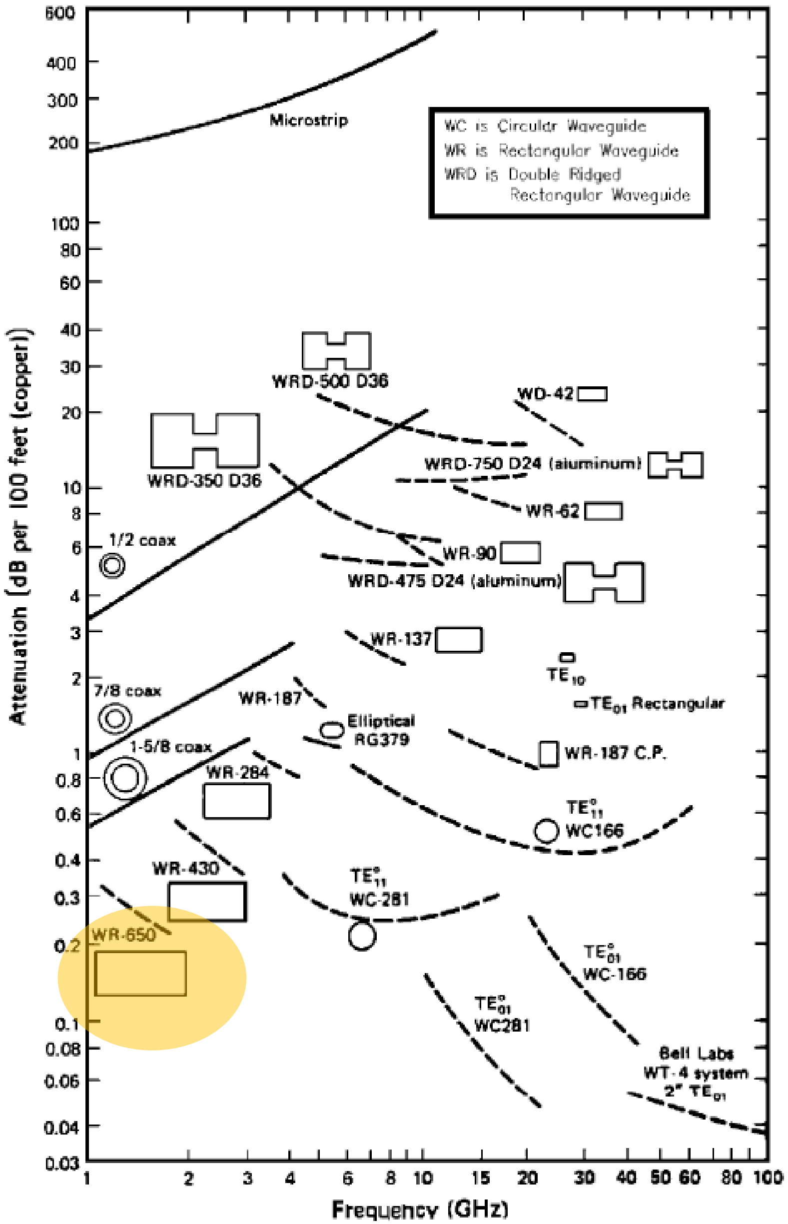
Attenuation of Various Transmission Lines



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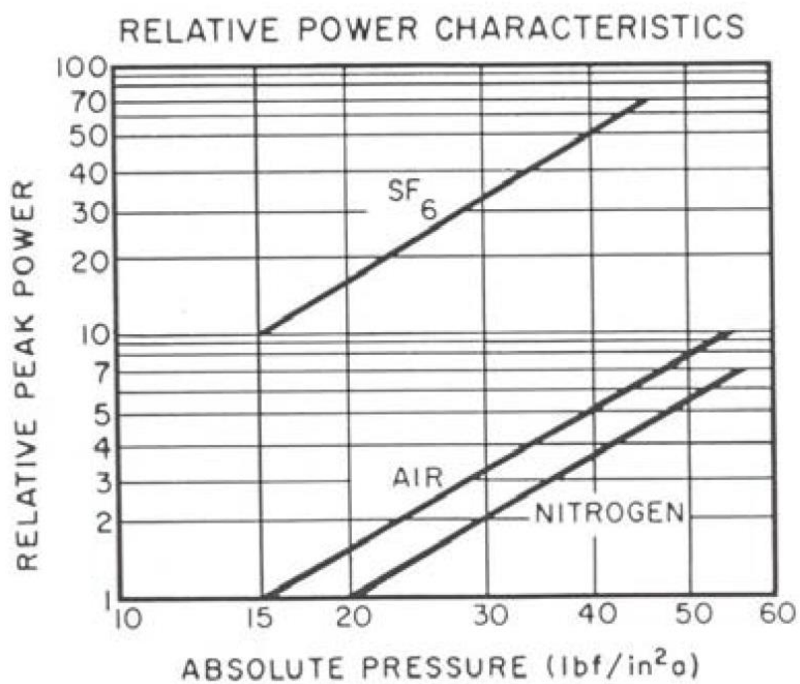
Example: WR650 Waveguide

- Standard WR650 rectangular waveguides (6.5” x 3
  used to power L-band accelerators
- For JLab’s cavities (f_{RF} = 1.497 MHz) the maximally
  WR650 waveguides is ~**65 MW** (far above the req
  upgrade cavities)
- What about attenuation?
- **Everything OK?**
- In reality: Reflections in waveguides such as arisin
  waveguide bends can reduce the breakdown field
- Other insertion devices can significantly reduce th
  theoretical waveguide limit



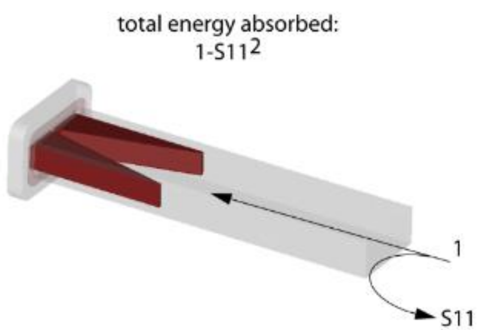
- Waveguides and coaxial lines are sometimes filled with dielectric sulfur hexafluoride (SF6) gas to increase the RF breakdown field limit (1 bar SF6 is equivalent to ~3 bar air), however SF6 is a potential greenhouse gas

https://www.megaind.com

Experiment with Rectangular Waveguide

- Rectangular waveguide with various terminations

1) Measure the reflection response S_{11} using coaxial-to-waveguide adapter using VNA
2) Make use of calibration kit (1-port calibration)
3) Learn how to de-embed a device under test (DUT) utilizing the Time Domain Reflectometry (TDR) option of the VNA
4) Measure the reflection response of the adapter only by setting appropriate time gates
5) What is the useful range for the measurements? Note: All adapters are bandwidth-limited and allow only for a certain frequency range to be transmitted efficiently
6) Measure the reflection response of the termination (HOM-loads) by re-adjusting the time gates
7) Record the reflection response and determine the characteristics for each HOM-load
8) How does using the TDR option compare to regular results?
9) What is the best HOM-load at room temperature?



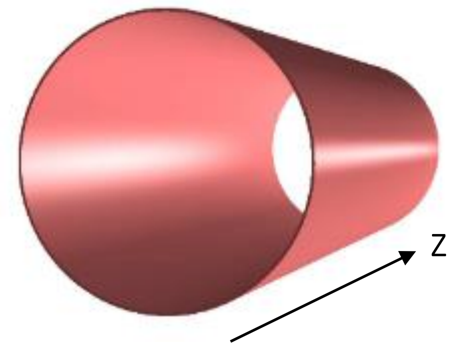
HOM damping waveguide dimensions:
H x B = 0.71" x 6.3" = 18 mm x 160 mm

Standard WR430 adapter:
H x B = 1.875" x 4.3"

http://www.agilent.com/

Appendix

Round Waveguide



R = inner radius

- We assume an infinitely long waveguide in z-direction
- An existing wave inside (previously launched from on side of the guide) can be propagating only in one direction (no reflection plane)
- Instead of an infinitely long waveguide one can also assume that the waveguide is perfectly matched on one end such that no reflection occurs
- Derivation of fields is analogous to cylindrical resonator, except for z-direction

$$\boxed{\vec{A} = N(r)M(\varphi)P(z)} \qquad \boxed{P(z) = Be^{\pm\gamma z}} \qquad \boxed{\gamma = \alpha + i\,\beta}$$

forwards or backwards travelling wave

- Longitudinal index p: Since there is no boundary in z-direction, a dependency on a number 'p' is undetermined
- Azimuthal index m: $\boxed{M(\varphi) = C_1 e^{-im\varphi} + C_2 e^{+im\varphi}}$ $\boxed{M(\varphi) = M(\varphi + 2\pi)}$ $; m \in \mathbb{N}^0$
- Radial index n: $\boxed{N(r) = D_1 J_m\left(r\sqrt{k^2+\gamma^2}\right)}$
- ***TM***-modes: $\boxed{E_z(r) \propto N(R) = D_1 J_m\left(R\sqrt{k^2+\gamma^2}\right) = 0}$ $\rightarrow$ $\boxed{x_{mn} = R\sqrt{k^2+\gamma^2}}$ $\rightarrow$ $\boxed{\gamma = \pm\, i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x_{mn}}{R}\right)^2}}$

  Tangential fields of E_z(r=R) must vanish:

  $;k^2 = \mu\epsilon\omega^2$

- ***TE***-modes: $\boxed{\frac{\delta H_z(r=R)}{\delta r} = 0}$ $\boxed{\frac{\partial N(r=R)}{\partial r} = D_1 \frac{\partial}{\partial r} J_m\left(R\sqrt{k^2+\gamma^2}\right) = 0}$

  E_z = 0, azimuthal fields E_φ(r=R) must vanish at cavity perimeter

  $\boxed{x'_{mn} = R\sqrt{k^2+\gamma^2}}$ $\rightarrow$ $\boxed{\gamma = \pm\, i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x'_{mn}}{R}\right)^2}}$

Round Waveguide

- Loss-less case ($\alpha = 0$)

$$\gamma^{TM} = i\beta = \pm i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x_{mn}}{R}\right)^2}$$

$$\gamma^{TE} = i\beta = \pm i\sqrt{\mu\epsilon\omega^2 - \left(\frac{x'_{mn}}{R}\right)^2}$$

- Only if β is real, the wave can propagate without decay by means of $e^{-i\beta z}$
- Leads to so-called **cutoff frequencies**, beyond which wave may propagate for given m, n and R

$$\mu\epsilon\omega^2 \geq \left(\frac{x_{mn}}{R}\right)^2$$

$$\mu\epsilon\omega^2 \geq \left(\frac{x'_{mn}}{R}\right)^2$$

- Cutoff-frequencies for *TM* (use x_{mn}) and *TE* modes (use x'_{mn}):

$$\omega_c^{TM,TE} = 2\pi f_c^{TM,TE} = 2\pi \frac{v}{\lambda_c^{TM,TE}} = \frac{1}{\sqrt{\mu\epsilon}} \cdot \left(\frac{x_{mn} \text{ or } x'_{mn}}{R}\right)$$

- The spectrum of possible modes above cutoff frequencies is continuous

- Inserting last in first equations on this page yields: $\beta^2 = \mu\epsilon(\omega^2 - \omega_c^2)$; *TE* or *TM* modes

Round Waveguide - Field Components

TM-Modes

$$E_r(\omega) = -E_0 \left(\frac{\pi R^2}{x_{mn}^2}\right) \frac{\partial J_m\left(r\frac{x_{mn}}{R}\right)}{\partial r} cos(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$E_\varphi(\omega) = E_0 \left(\frac{m}{r}\right)\cdot\left(\frac{\pi R^2}{L\, x_{mn}^2}\right)\cdot J_m\left(r\frac{x_{mn}}{R}\right)\cdot sin(m\varphi)\, e^{-i\beta z}\cdot e^{i\omega t}$$

$$E_z(\omega) = E_0 \cdot J_m\left(r\frac{x_{mn}}{R}\right)\cdot cos(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$H_r(\omega) = -iE_0 \cdot \left(\frac{m}{r}\right)\cdot\left(\frac{\omega^{TM}\varepsilon R^2}{x_{mn}^2}\right)\cdot J_m\left(r\frac{x_{mn}}{R}\right)\cdot sin(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$H_\varphi(\omega) = -iE_0 \cdot \left(\frac{\omega^{TM}\varepsilon R^2}{x_{mn}^2}\right)\cdot \frac{\partial J_m\left(r\frac{x_{mn}}{R}\right)}{\partial r}\cdot cos(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$H_z(\omega) = 0$$

$$\omega_c^{TM} = \frac{1}{\sqrt{\mu\epsilon}}\cdot\left(\frac{x_{mn}}{R}\right)$$

TE-Modes

$$E_r(\omega) = iH_0 \cdot \left(\frac{m}{r}\right)\cdot\left(\frac{\omega^{TE}\mu R^2}{(x'_{mn})^2}\right)\cdot J_m\left(r\frac{x'_{mn}}{R}\right)\cdot sin(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$E_\varphi(\omega) = iH_0 \cdot \left(\frac{\omega^{TE}\mu R^2}{(x'_{mn})^2}\right)\cdot \frac{\partial J_m\left(r\frac{x'_{mn}}{R}\right)}{\partial r}\cdot cos(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$E_z(\omega) = 0$$

$$H_r(\omega) = H_0 \cdot \left(\frac{\pi R^2}{(x'_{mn})^2}\right)\cdot \frac{\partial J_m\left(r\frac{x'_{mn}}{R}\right)}{\partial r}\cdot cos(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$H_\varphi(\omega) = -H_0 \cdot \left(\frac{m}{r}\right)\cdot\left(\frac{\pi R^2}{(x'_{mn})^2}\right)\cdot J_m\left(r\frac{x'_{mn}}{R}\right)\cdot sin(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

$$H_z(\omega) = H_0 \cdot J_m\left(r\frac{x'_{mn}}{R}\right)\cdot cos(m\varphi)\cdot e^{-i\beta z}\cdot e^{i\omega t}$$

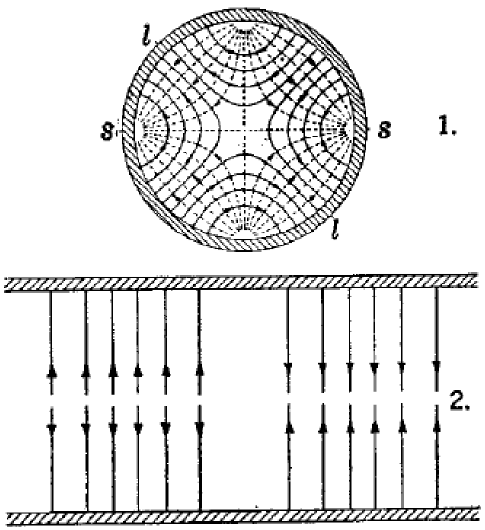
$$\omega_c^{TE} = \frac{1}{\sqrt{\mu\epsilon}}\cdot\left(\frac{x'_{mn}}{R}\right)$$

Note: Cutoff frequency determined by interior medium (permittivity, permeability), tube radius (R), and roots (x_{mn}, x'_{mn}) of Bessel function of first kind (*TM*-modes) or its derivative (*TE*-modes)

TE-Mode Pattern

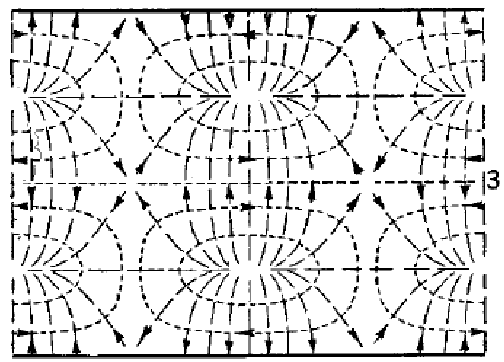
1. Cross-sectional view
2. Longitudinal view through plane *l-l*
3. Surface view from *s-s*

TE$_{21}$



H_{21}

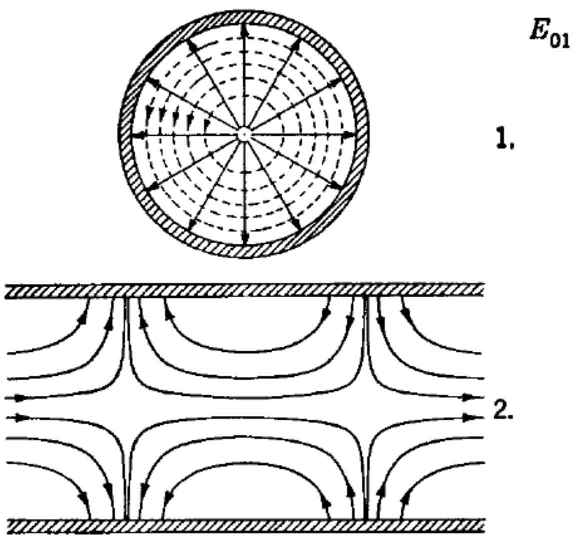
$\lambda_c = 2.057\ a$

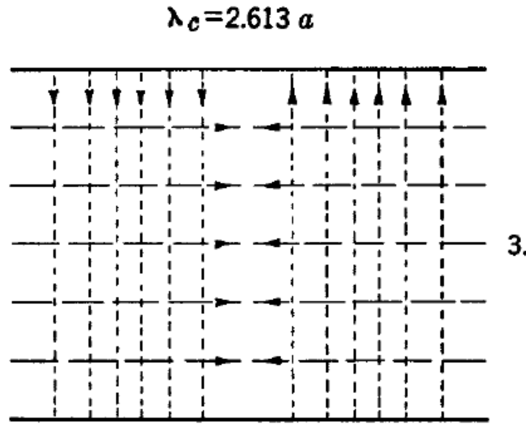


TM-Mode Pattern

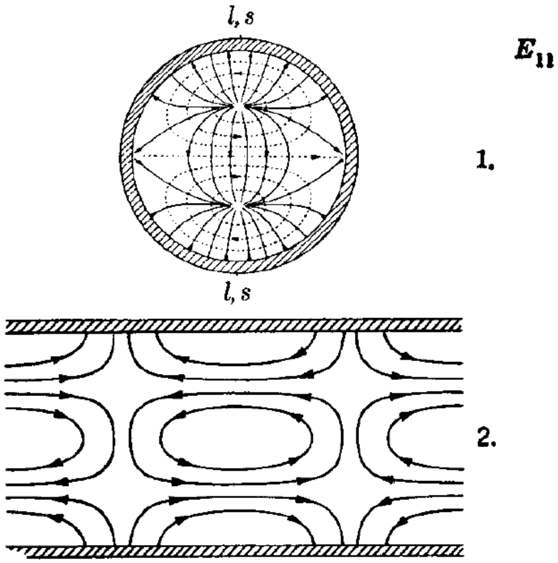
1. Cross-sectional view
2. Longitudinal view through plane *l-l*
3. Surface view from *s-s*

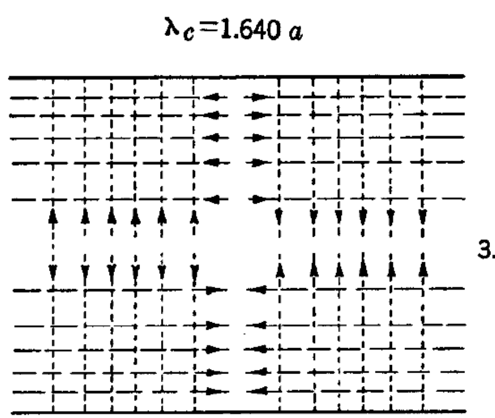
TM$_{01}$





TM$_{11}$





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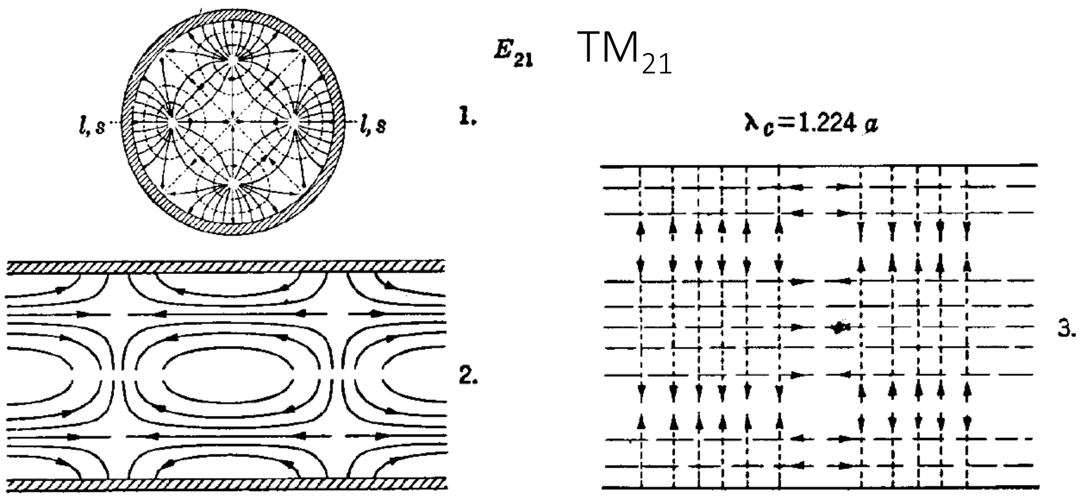
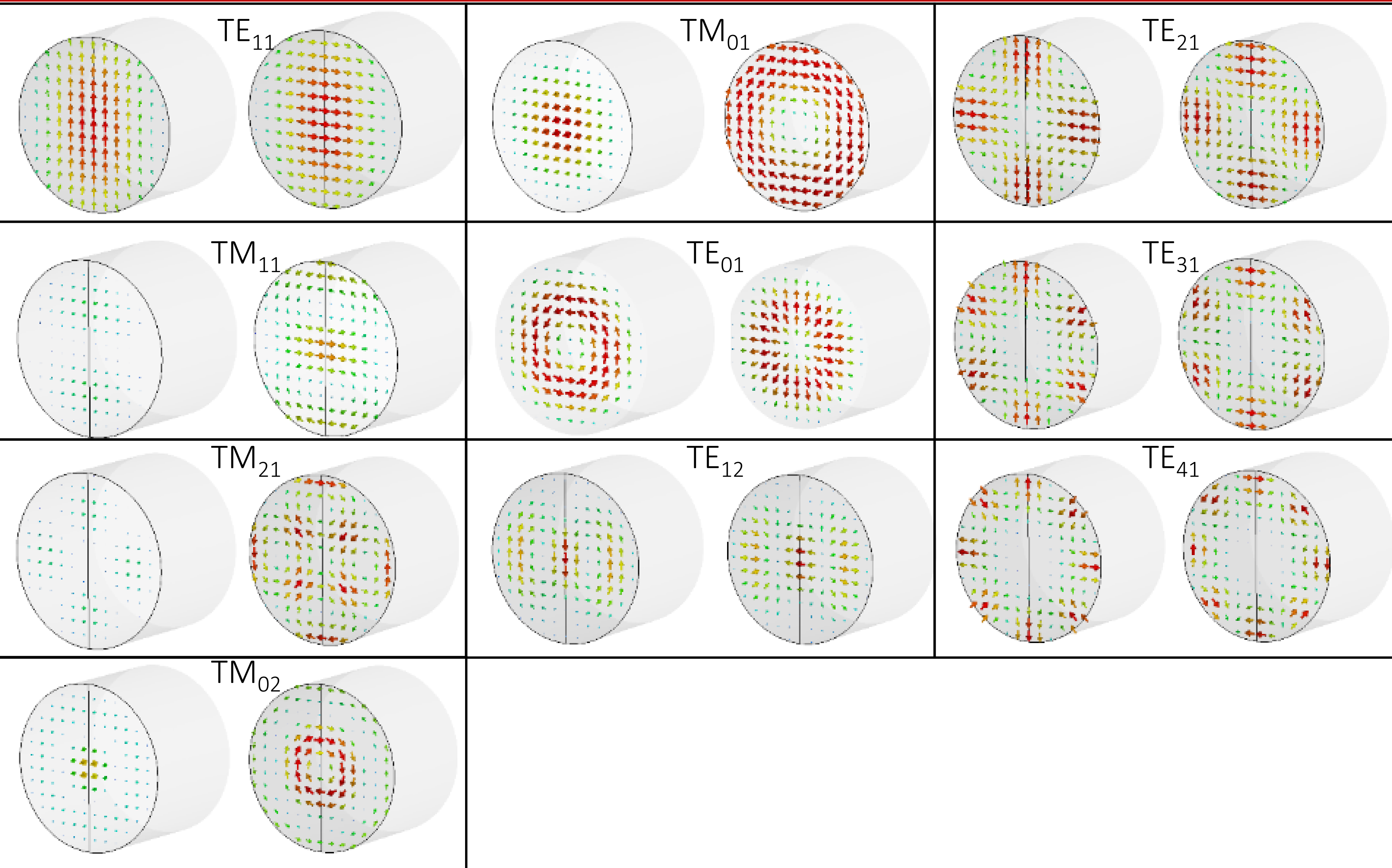
TM-Mode Pattern

TM$_{21}$



1. Cross-sectional view
2. Longitudinal view through plane *l*-*l*
3. Surface view from *s*-*s*

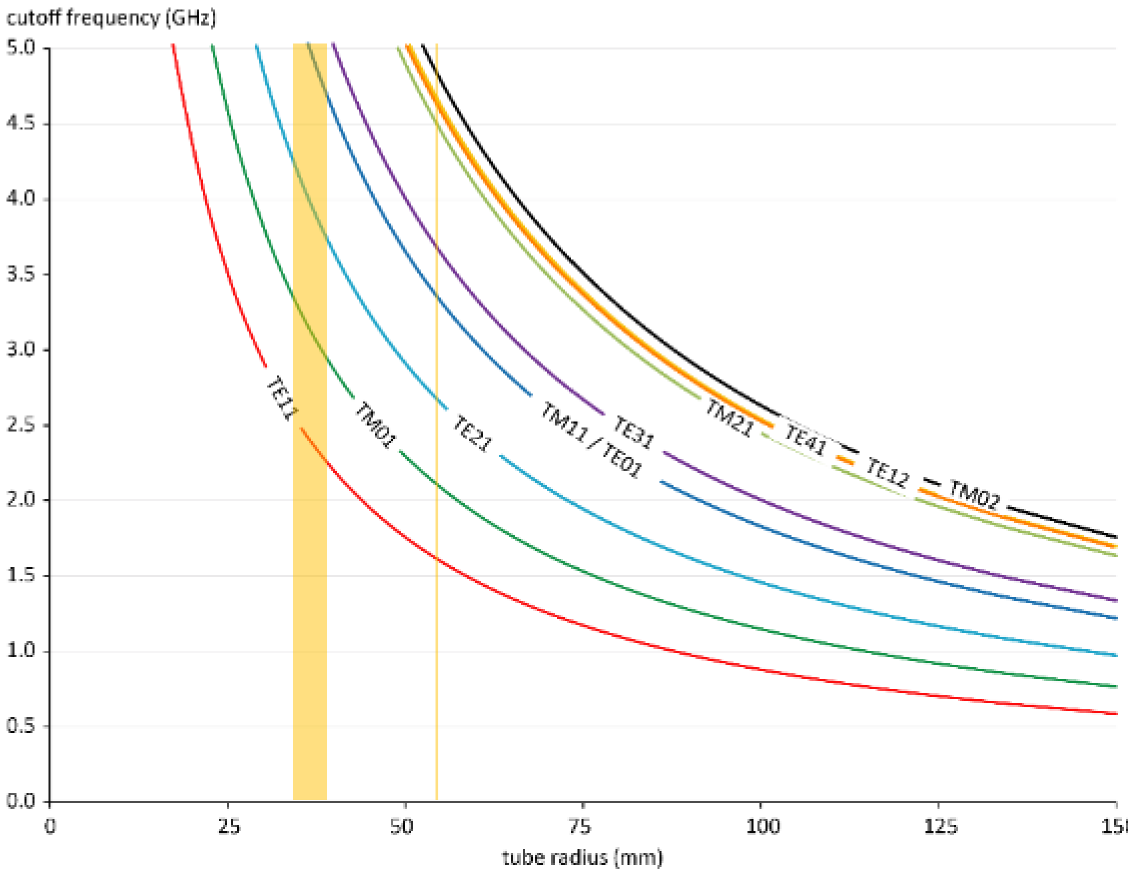
First 10 Beam Tube Modes (E/H-fields)

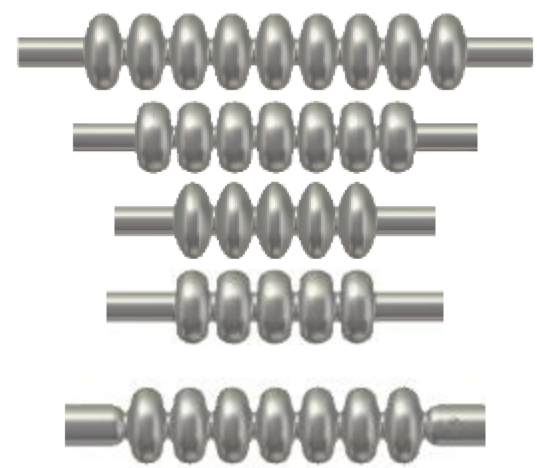


Cutoff Frequencies of Cavity Beam Tubes

- R = 35-39 mm are typical tube radii for 1.3-1.5 GHz (L-band) SRF cavities, e.g. EU-XFEL/ILC/LCLS-II TESLA-type cavities or JLab's CEBAF/FEL cavities

- Larger tube radii are considered for high-current heavily HOM-damped cavities, which lets Higher Order Modes propagate out of cavity at lower frequencies, e.g. 55 mm for Cornell 1.3 GHz ERL cavity design





Jefferson Lab

Rectangular Waveguide - Field Components

TM-Modes

$$E_x(\omega) = -iE_0 \cdot \frac{\beta}{k_c^2} \cdot \frac{m\pi}{a} \cdot cos\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot sin\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$E_y(\omega) = -iE_0 \cdot \frac{\beta}{k_c^2} \cdot \frac{n\pi}{b} \cdot sin\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot cos\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$E_z(\omega) = E_0 \cdot sin\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot sin\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$H_x(\omega) = iE_0 \cdot \frac{\omega\cdot\varepsilon}{k_c^2} \cdot \frac{n\pi}{b} \cdot sin\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot cos\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$H_y(\omega) = -iE_0 \cdot \frac{\omega\cdot\varepsilon}{k_c^2} \cdot \frac{m\pi}{a} \cdot cos\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot sin\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$H_z(\omega) = 0$$

n, m **≥ 1** (m or n = 0 → $E_z(\omega) = 0$)
TM_{11} is 1st propagating *TM*-mode

$$E_x = Z_{TM} \cdot H_y$$

$$E_y = -Z_{TM} \cdot H_x$$

TE-Modes

$$E_x(\omega) = iH_0 \cdot \frac{\omega\cdot\mu}{k_c^2} \cdot \frac{n\pi}{b} \cdot cos\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot sin\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$E_y(\omega) = -iH_0 \cdot \frac{\omega\cdot\mu}{k_c^2} \cdot \frac{m\pi}{a} \cdot sin\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot cos\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$E_z(\omega) = 0$$

$$H_x(\omega) = iH_0 \cdot \frac{\beta}{k_c^2} \cdot \frac{m\pi}{a} \cdot sin\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot cos\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$H_y(\omega) = iH_0 \cdot \frac{\beta}{k_c^2} \cdot \frac{n\pi}{b} \cdot cos\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot sin\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

$$H_z(\omega) = H_0 \cdot cos\left(\frac{m\cdot\pi\cdot x}{a}\right) \cdot cos\left(\frac{n\cdot\pi\cdot y}{b}\right) \cdot e^{-i\beta z} \cdot e^{i\omega t}$$

n, m **≥ 0** (n=m=0 not allowed, $H_z(\omega)$ static)
TE_{10} is first propagating *TE*-mode

$$E_x = Z_{TE} \cdot H_y$$

$$E_y = -Z_{TE} \cdot H_x$$

$$k_c^2 = \left(\frac{m\cdot\pi}{a}\right)^2 + \left(\frac{n\cdot\pi}{b}\right)^2$$

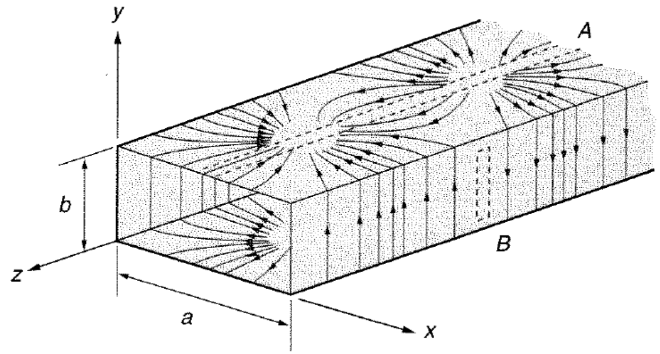
$$\omega_c = 2\pi f_c = \frac{\pi}{\sqrt{\mu\epsilon}} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

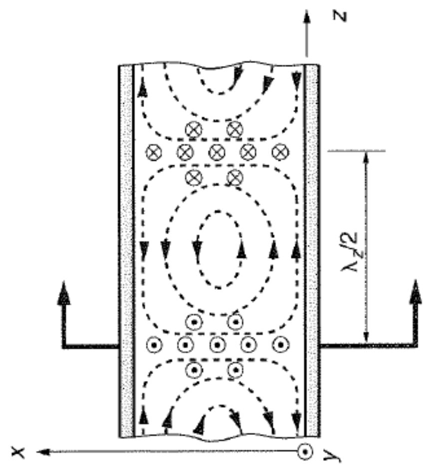
$$\beta = \sqrt{\mu\epsilon}\omega \cdot \sqrt{1 - \frac{f_c{}^2}{f^2}}$$

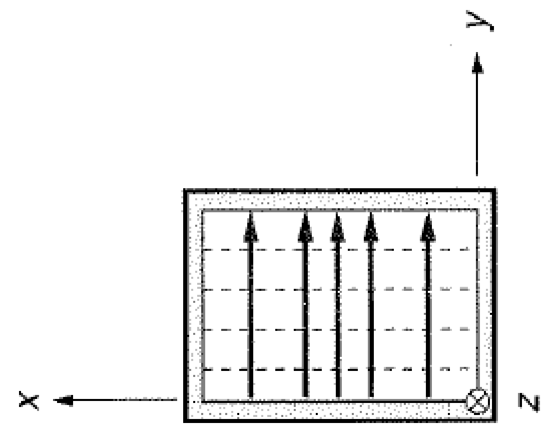
Note: Cutoff frequency determined by interior medium (permittivity, permeability) and waveguide internal dimensions a and b (a = inner width, b = inner height)

Mode Pattern *TE*- and *TM*-mode, respectively

TE_{10}



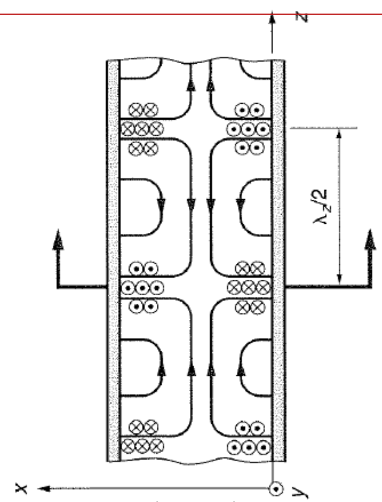


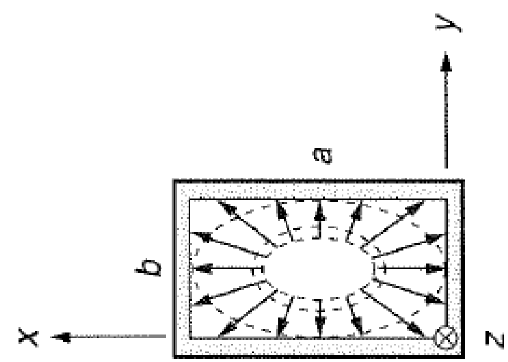


Surface current distribution for the TE_{10} mode at a fixed time. The mode moves in z direction.

E-field (solid) lines and H-field (dashed) lines at a fixed time.

TM_{11}





E-field (solid) lines and H-field (dashed) lines at a fixed time.

These figures are taken from a textbook from Prof. Z. Popovic, 'Electromagnetics Around Us: Some Basic Concepts'

Mode Pattern